ANALYTICAL METHODS FOR LÉVY PROCESSES WITH APPLICATIONS TO FINANCE

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Abstract

This dissertation is divided into two parts: the first part is a literature review and the second describes three new contributions to the literature. The literature review aims to provide a self-contained introduction to some popular Lévy models and to two key objects from the theory of Lévy processes: the Wiener-Hopf factors and the exponential functional. We pay special attention to techniques and results associated with two "analytically tractable" families of processes known as the meromorphic and hyper-exponential families. We also demonstrate some important numerical techniques for working with these families and for solving numerical integration and rational approximation problems.

In the second part of the dissertation we prove that the exponential functional of a meromorphic Lévy process is distributed like an infinite product of independent Beta random variables. We also identify the Mellin transform of the exponential functional, and then, under the assumption that the log-stock price follows a meromorphic process, we use this to develop a fast and accurate algorithm for pricing continuously monitored, fixed strike Asian call options. Next, we answer an open question about the density of the supremum of an α-stable process. We find that the density has a conditionally convergent double series representation when α is an irrational number. Lastly, we develop an effective and simple algorithm for approximating any process in the class of completely monotone processes – some members of this class include the popular variance gamma, CGMY, and normal inverse Gaussian processes – by a hyper-exponential process. Under the assumption that the log-stock price follows a variance gamma or CGMY process we use this approximation to price several exotic options such as Asian and barrier options. Our algorithms are easy to implement and produce accurate prices.

Für Petra und Thomas Hackmann.

Acknowledgements

Before attending graduate school I often heard some version of the following maxim from former graduate students, "your experience, good or bad, will depend largely on your supervisor." I am happy to say that I had a good experience and that the maxim held true for me. I attribute a large part of my good experience, and of my relative success, to my supervisor Alexey Kuznetsov. He was instrumental in introducing me to the world of mathematical research and without his support I could not have written this dissertation. Beyond sharing his knowledge of probability, mathematical finance, complex analysis, numerical analysis, and special functions with me, he also taught me something about the importance of intuition and how to conduct myself in the world of academia. For all of these reasons I am very grateful and thankful. Thank you, Alexey.

After supporting me in my initial career choice, my parents did not hesitate to support me again when I announced my somewhat unusual desire to stop working and study mathematics. Without their encouragement and financial support I could not have finished this project; this dissertation is rightfully dedicated to them. Throughout my studies I also took comfort in speaking with my brother Jonas about the ups and downs of life as a Ph.D. student and I felt the support of his family. Ein herzliches Dankeschön an meine Familie.

I wrote this dissertation in Győr, Hungary where my girlfriend Ági made room for me in her home. Throughout the process she listened patiently to my complaints and worries and did not make her own complaints when I worked late or on the weekend. I thank her for her patience and love. Köszönöm és szeretlek, Mackó.

Contents

Chapter 1

Introduction

1.1 General overview and introduction

It is rare that we solve problems in probability by making purely probabilistic arguments. Rather, we often rely on connections with other mathematical theory. For example, we might use ideas from combinatorics, partial differential equations, and complex analysis. Appealing to techniques from outside probability makes certain proofs possible; likewise, probability theory is employed in other fields to make advances.

In this dissertation we employ and develop complex analytic techniques to solve problems involving Lévy processes. Lévy processes are important stochastic processes: they are general enough to represent real-world phenomena, and yet tractable enough to allow for meaningful analysis. They appear in many applications in the natural and physical sciences and are used extensively in actuarial science and mathematical finance. The chief application in this work is mathematical finance, although the techniques and theorems described here are not limited to this purpose.

The problems we consider involve either the extrema of a Lévy process, or the average value of an exponentiated Lévy process. Specifically, we are interested in the distributions of the extrema processes and in pricing exotic financial products like barrier and Asian options. Theoretical solutions to our problems have already been derived by other authors: problems involving the extrema can be solved by building a connection to the Wiener-Hopf factorization, while problems involving the average value can be solved by making a connection with the exponential functional.

Both the Wiener-Hopf factors and the exponential functional are well-studied and important

objects from the theory of Lévy processes. Additionally, they appear in connection with the solutions of many applied problems. In fact, finding an explicit expression for the Wiener-Hopf factors or the distribution of the exponential functional – or even some transform of the distribution – is often sufficient to solve applied problems in a satisfactory way. This is the case for the problems in this dissertation. Unfortunately, explicit expressions are the exception rather than the rule, therefore, much of this dissertation is devoted to methods for calculating or approximating the Wiener-Hopf factors and the distribution of the exponential functional for specific Lévy processes.

By "approximating" we mean that we usually need to resort to numerical means to complete our calculations; this is unavoidable. However, we also mean that we substitute "analytically tractable" processes for processes which are more difficult to work with. The critical hurdle in this substitution process is to ensure that the process we substitute behaves like the original process. For example, problems from mathematical finance typically demand that we work with infinite activity processes. Therefore, in this scenario, we either need to find an analytically tractable process which has infinite activity, or we need to suitably match the behaviour of a finite activity process.

As already mentioned, the primary mathematical tools employed in this dissertation are from complex analysis. We work primarily with integral transforms (Fourier, Laplace, Mellin) to take probabilistic objects for which we do not have explicit expressions into the complex plane. Here we use the methods of classical analysis to obtain a tractable expression for the transformed object, and analytically or numerically invert the transform to get our result. By "analytically tractable" processes we mean precisely those families of processes which are easy to work with under such transforms. For example, the Laplace exponent of a hyper-exponential process is a simple rational function with real poles and residues. This makes hyper-exponential processes easy to work with and allows us to derive nearly explicit solutions to many problems.

The two work-horse analytically tractable families employed in this dissertation are precisely the family of hyper-exponential processes and its generalization, the family of meromorphic processes. The latter of these is relatively new to the literature having been introduced in 2010 in [65]. Generally speaking, the primary purpose of this dissertation is to demonstrate existing and new techniques for working with these families, specifically for problems arising in mathematical finance. These problems invariably involve the two key theoretical objects, namely, the Wiener-Hopf factors and the exponential functional.

A secondary purpose is to study the family of α-stable (or just stable) processes, which also enjoys a degree of analytical tractability. This tractability is inherited from the stable process' self-similarity property, which, roughly speaking, means that scaling a stable process in space is equivalent to scaling in time. We review the work of a list of authors – Darling [36], Heyde [57], Doney [38], and Kuznetsov [68] – who have incrementally determined the Wiener-Hopf factors for stable processes for a wider and wider array of possibilities for the parameter α. We demonstrate how the analytical properties of the Wiener-Hopf factors lead us directly to an expression for the density of the supremum process (this is the recent work of Hubalek and Kuznetsov [58] and Kuznetsov [70]) and answer an (in some sense final) open question about this density.

1.2 Detailed overview and summary of specific results

This dissertation is split into two parts: Part I is intended as a literature review, and to introduce concepts, and analytical and numerical techniques which are used in Part II to derive new results. There are practically no new results in Part I (save for some discussion and Theorem 23 in Section 5.3) but the majority of definitions are made here. Part II consists of three chapters each corresponding to a new contribution to the literature; see the next section for publication and authorship details, and the summaries below for details of each contribution.

Part I : Literature review and overview of techniques

Chapter 2: General notation and some families of Lévy processes

Here we establish some notation, definitions, and conventions which are used throughout the text. We also briefly introduce Lévy processes. Most importantly we introduce, in some detail, the families of Lévy processes which are used in the dissertation. We list Lévy densities and characteristic/Laplace exponents, and give specific examples of particular important processes. We also mention any important applications, especially connections to mathematical finance, if these exist.

Chapter 3: The Wiener-Hopf factorization

We state the collection of theorems and identities known as the Wiener-Hopf factorization for Lévy processes, and give some intuition of their proof by sketching the ideas for the simpler case of the random walk. Further, we list those processes (except the most basic

cases, e.g. Brownian motion) for which we have explicit expressions of the Wiener-Hopf factors, and give these expressions. Of these, the results on stable processes will inform our work in Chapter 7, those on meromorphic processes will inform our work in Chapter 6 and those on completely monotone processes and hyper-exponential processes will inform our work in Chapter 8. Importantly, we discuss three existing techniques, which to the author's knowledge forms an exhaustive list, of known methods for deriving such expressions. Lastly, we give two examples of applications of the Wiener-Hopf factorization in finance, the most important being that of pricing barrier options which we see again in Chapter 8.

Chapter 4: The exponential functional

The primary purpose of this chapter is to introduce the exponential functional, and a verification result for determining its Mellin transform. We demonstrate how to apply this result for hyper-exponential processes and processes with jumps of rational transform. This serves as the inspiration for our work in Chapter 6 where we use the same technique to determine the distribution of the exponential functional for meromorphic processes. Additionally, we demonstrate a number of different ways in which the exponential functional can be used to price Asian options; one of these is used again in Chapter 6. Lastly, no discussion of the exponential functional would be complete without some mention of the Lamperti transform and the connection with positive self-similar Markov processes. We demonstrate how the exponential functional and the Lamperti transform can be used to derive an expression for the density of the supremum of a stable process – our focus in Chapter 7.

Chapter 5: Numerical techniques

As mentioned in the previous section, avoiding numerical methods is not possible for the problems we hope to solve. The aim of this chapter is three-fold: the first aim is to discuss numerical methods for evaluating oscillatory integrals like those we might encounter when inverting integral transforms; the second aim is to discuss some numerical issues that are particular to our work-horse meromorphic and hyper-exponential families; and the third aim is to discuss methods of rational approximation for two particular classes of analytic functions. This latter discussion is the most important piece of the chapter. We introduce the interesting and useful connection between processes with completely monotone jumps, Pick functions, and Stieltjes functions, as well as rational approximations of these functions. Further, we begin to explain how these connections can help us build an algorithm for approximating *any* completely monotone process by a hyper-exponential process. This discussion is continued in Chapter 8 where its full consequences are revealed.

Part II : New results

Chapter 6: Asian options and meromorphic Lévy processes

We consider the price of a continuously sampled arithmetic rate Asian option with fixed maturity. In mathematical notation, we are interested in determining the following expectation,

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(A_0 \int_0^T e^{X_u} \mathrm{d}u - K\right)^+\right],$$

where A_0 is the initial price of the underlying security, K is the strike price, T is the expiry time, $z^+ = \max\{z, 0\}$, and X is a stochastic process. Deriving a manageable formula for C is difficult because of the "path dependent" nature of Asian options; specifically, the expression $Z_t = A_0 \int_0^t e^{X_u} \mathrm{d}u$ is not a Markov process.

The pricing problem has been approached in numerous settings for the process X: see, for example, [87, 105] for Brownian motion, [110] for jump diffusions, and [12] for the general semimartingale case. Despite the prevalence of research in the area, the author is not aware of any explicit or semi-explicit pricing formulas for processes with two-sided jumps and infinite jump activity. This is significant, as recent research [3, 29] shows that certain stock prices are best modeled by processes with precisely these properties.

In this chapter we derive a formula for the Mellin-Laplace transform of C for the case where X is a meromorphic process. Here it is important to note that meromorphic processes can exhibit all manner of jump behaviour: two-sided jumps; infinite activity jumps; and jumps resulting in paths having infinite total variation. We then develop a fast and accurate numerical procedure to recover C. Specifically, we express the Mellin-Laplace transform of C in terms of the Mellin transform of the exponential functional. The resulting expression is well-suited for numerical inversion techniques of Chapter 5.

The inspiration for the Mellin transform approach is taken from an article by Cai and Kou [27] who develop a similar algorithm for hyper-exponential processes –this is discussed in Chapter 4. Despite the heightened complexity of our problem we find our pricing algorithm performs at least as well as theirs.

In deriving the algorithm, we also establish a valuable theoretical result of independent interest. Using the verification result of Chapter 4 we derive the Mellin transform of the

exponential functional for a general meromorphic process. We subsequently find that the distribution of the exponential functional is that of an infinite product of Beta random variables. To the best of the author's knowledge, this is the first instance in which the distribution of the exponential functional is known (in full generality) for a process with double-sided, infinite activity jumps and infinite variation paths.

Chapter 7: The density of the supremum of a stable process

Finding a formula for the density $p(x)$ of the supremum of an α-stable process is a challenging problem that has been studied by mathematicians since the 1950's. Recently there has been significant progress towards a solution. Hubalek and Kuznetsov [58] have derived an absolutely convergent double series representation for $p(x)$ for processes whose scaling parameters α are in $\mathbb{R}\backslash(\mathbb{Q}\cup\mathcal{L})$. Here $\mathcal{L}$ is a subset of the irrational numbers which is small in the sense that it has Lebesgue measure and Hausdorff dimension equal to zero. Their derivation is based on the Mellin transform of the positive Wiener-Hopf factor, which can be calculated explicitly owing to the analytical properties of the Wiener-Hopf factors for stable processes (discussed in Chapter 3). Although this is an important result, it is also somewhat unsatisfactory: Kuznetsov [70] shows that for a subset of $\mathcal{L}$ the double series representation is not absolutely convergent, nor is it clear that there is a conditionally convergent alternative. Ideally, of course, we would like to find an expression that is valid for *all* irrational α.

In this chapter we find this conditionally convergent double series by making a small, but non-trivial change in the proof in [70]. We show that for all irrational α, the density $p(x)$ may be represented by the double series from [70] provided the series is summed in a way that depends on the arithmetic properties of α. Specifically, we show the existence of an increasing sequence $\{c_k\}_{k\geq 1}$ whose terms depend on α, such that when we sum the double series over the triangles $\{(m,n) : m,\, n \geq 0,\, 0 \leq m+\alpha n < c_k\}_{k\geq 1}$ it converges for any irrational α.

Chapter 8: Approximating Lévy processes with completely monotone jumps

The variance gamma (VG), normal inverse Gaussian (NIG), and CGMY processes are all examples of popular infinite activity, completely monotone processes used in mathematical finance. However, in Chapter 3 we see that none have explicit Wiener-Hopf factorizations, nor can we identify the distribution of the exponential functional. This means that the pric-

ing of certain financial products like barrier and Asian options is not possible. On the other hand, we have an explicit expression for the Wiener-Hopf factors of a hyper-exponential process (Chapter 3) and we know the distribution of the exponential functional (Chapter 4). This means that we can use the methods of Chapter 3 to price barrier options and the methods of Chapter 4 to price Asian options. Unfortunately, hyper-exponential processes are not infinite activity processes, and thus are not necessarily appropriate for modeling stock prices. This leads to a trade-off: we can use the popular more realistic models which are difficult to work with, or we can use hyper-exponential processes which are easy to manipulate.

In this paper, we eliminate the trade-off between model fit and tractability by developing a simple method for approximating VG, CGMY, and NIG processes with hyper-exponential processes. In fact, this procedure works for the entire class of processes with completely monotone jumps. We approximate the Laplace exponent of any such process by a rational function $\psi_n(z)$ where n is the degree of the numerator polynomial. We prove that any rational function obtained by our method: a) is the Laplace exponent of hyper-exponential process; and b) matches a maximum number of moments of the original process. Further, as $n \to \infty$ our approximations converge uniformly and exponentially fast to the Laplace exponent of the original process on compact sets in the cut complex plane. Finally, we show how to use our method to solve several option pricing problems where the stock prices are determined by either VG or CGMY processes. Notably, we develop a fast and simple algorithm for pricing down-and-out barrier options and show that when n is as small as 6 we match benchmark prices to the cent.

Our approximation method is based on the interesting connection between completely monotone processes, Stieltjes functions, and Padé approximants (rational approximations) which is first discussed in Chapter 5. In Chapter 8 we fully exploit this connection, and further connections with other topics from classical analysis such as the Gaussian quadrature, and orthogonal polynomials.

1.3 Publication details

The contents of Chapters 6, 7, and 8 have either been published or been accepted for publication. The results appearing in these chapters represent joint work with Alexey Kuznetsov. A modified version of: Chapter 6 has appeared in *Finance and Stochastics* [55]; Chapter 7 has appeared in *Electronic Communications in Probability* [53]; Chapter 8 is forthcoming in *The Annals of Applied Probability* [54].

Part I

Literature review and overview of techniques

Chapter 2

General notation and some families of Lévy processes

In this chapter, we first establish some common notation and recall some fundamental properties of Lévy processes. Then, we present a collection of Lévy processes to which we will refer throughout the work. We describe their relevant features, provide references for further reading and state any relevant applications.

2.1 Notation, assumptions, and basic information

2.1.1 General

- All definitions are given in ***bold italic*** type and are presented inline in the text. The location of any definition within this dissertation is cataloged in the Index.
- There are a number of symbols which consistently refer to the same objects throughout the entire text. These are cataloged in the List of symbols.
- We refer to a number of special functions in the text without necessarily defining them there. Should readers encounter an unfamiliar function, they are advised to consult Appendix A where they will find a definition and any relevant identities.
- For the open and closed half line we will write:

$$\mathbb{R}^+ := (0, \infty),\ \bar{\mathbb{R}}^+ := [0, \infty),\ \mathbb{R}^- := (-\infty, 0),\ \bar{\mathbb{R}}^- := (-\infty, 0].$$

- Similarly for the half plane:

$$\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},\ \bar{\mathbb{C}}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\},$$
$$\mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\},\ \bar{\mathbb{C}}^- := \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}.$$

- For $d \in \mathbb{N}$ we will denote $\mathcal{B}_{\mathbb{R}^d}$ as the Borel σ-algebra on $\mathbb{R}^d$. In general, for any topological space S we will write $\mathcal{B}_S$ for the Borel sigma algebra on S.

- In this work we will almost always use the principal branch of the logarithm, which is defined in the domain $|\arg(z)| < \pi$ by requiring that $\log(1) = 0$. Similarly, the power function will be defined as $z^a = \exp(a \log(z))$ in the domain $|\arg(z)| < \pi$.

- We will use the abbreviation i.i.d. for "independent and identically distributed" and CDF for "cumulative distribution function".

- For two random variables, ξ and ζ, we will write equality in distribution as $\xi \overset{d}{=} \zeta$.

- For a probability measure $\mu(\mathrm{d}x)$ we will write an event holds μ-a.s. if it holds μ almost surely. When there is no ambiguity as to the measure, we will simply write the event holds a.s..

- The support of a Borel measure $\mu(\mathrm{d}x)$ will be denoted $\mathrm{supp}(\mu)$.

- $\mathbb{I}(x \in A)$ is the indicator function of the set A.

- δ_x is the Dirac-delta measure at the point x, i.e. $\delta_x(A) := \mathbb{I}(x \in A)$ for a fixed x and any measurable set A.

- $\delta_{x,y}$ is the function $\delta_{x,y} := \mathbb{I}(x \in \{y\}) = \mathbb{I}(y \in \{x\})$.

- The notation $\mathrm{sgn}(z)$ is reserved for the sign function.

2.1.2 Stochastic processes and filtrations

For most of the material presented, it is enough to define a ***stochastic process*** as a collection of $\mathbb{R}$-valued random variables $X := \{X_t : t \geq 0\}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Occasionally, we will also consider $\mathbb{R}^d$, $d \in \mathbb{N}$, or $\mathbb{R}^+$-valued stochastic processes. In some cases, we may change the index set such that $t \in \mathbb{N} \cup \{0\}$ to obtain a discrete time process. The reader may assume the original definition whenever we refer to a stochastic process without stating a specific index set or state space.

For the remainder of Section 2.1.2 let I be an index set representing either $\mathbb{R}^+$ or $\mathbb{N} \cup \{0\}$, and let S be a state space representing either $\mathbb{R}^d$ or $\mathbb{R}^+$. We say two S-valued stochastic processes, $X = \{X_t : t \in I\}$ and $Y = \{Y_t : t \in I\}$, defined on spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ respectively, are ***equal in distribution***, and write $X \stackrel{d}{=} Y$ if

$$\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \bar{\mathbb{P}}(Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n)$$

for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$, and $A_1, \ldots, A_n \in \mathcal{B}_S$.

We sometimes need to consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} := \{\mathcal{F}_t : t \in I\}$ is a ***filtration*** , i.e. an increasing sequence of σ-algebras. Whenever we consider a stochastic process X on a filtered space, we always assume that $\mathbb{F}$ has been generated by the process. That is, we assume $\mathcal{F}_t$ is generated by $\{X_s\colon\ s \leq t\}$ and the null sets of $\mathbb{P}$. This filtration is called the ***natural filtration*** of the process. For continuous time processes we also assume the natural filtration is right continuous, that is,

$$\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}.$$

Finally, a random variable of the form $\tau : \Omega \to I$ is a ***stopping time*** for a stochastic process with natural filtration $\mathbb{F}$, if for any $t \in I$ the event $\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable.

2.1.3 Infinitely divisible random variables and Lévy processes

A random variable ξ is ***infinitely divisible*** if for each $n \in \mathbb{N}$, there exist n i.i.d. random variables $\{\xi_i\}$ such that

$$\xi \stackrel{d}{=} \xi_1 + \ldots + \xi_n.$$

This is equivalent to saying that for any $n \in \mathbb{N}$, the distribution of ξ is the convolution of n identical distributions. By the famous Lévy-Khintchine Formula (see Theorem 1.3 in [76]) this is again equivalent to the statement that there exists a unique triple (a, σ^2, Π) such that

$$\mathbb{E}[e^{iz\xi}] = e^{-\psi(z)}, \quad z \in \mathbb{R},$$

where,

$$\Psi(z) = \frac{\sigma^2 z^2}{2} - iaz - \int_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izx\mathbb{I}(|x| < 1))\Pi(\mathrm{d}x), \tag{2.1}$$

and where, $a \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\Pi(\mathrm{d}x)$ is a measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\Pi(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}\backslash\{0\}} \min(1, x^2)\Pi(\mathrm{d}x) < \infty. \tag{2.2}$$

The function $\Psi(z)$ is called the ***characteristic exponent*** of ξ.

The principal mathematical object in this dissertation is the ***Lévy process***, which is an $\mathbb{R}$-valued stochastic process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that possesses the following properties:

(i) The paths of X are right continuous with left limits $\mathbb{P}$-a.s.

(ii) $X_0 = 0$ $\mathbb{P}$-a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

Given a Lévy process X we may verify using properties (iii) and (iv) from the definition, that for any $n \in \mathbb{N}$ we may write

$$X_t = \left(X_t - X_{\frac{n-1}{n}t}\right) + \left(X_{\frac{n-1}{n}t} - X_{\frac{n-2}{n}t}\right) + \cdots + \left(X_{\frac{2}{n}t} - X_{\frac{t}{n}}\right) + X_{\frac{t}{n}} \stackrel{d}{=} X_{\frac{t}{n},1} + \ldots + X_{\frac{t}{n},n},$$

where the $X_{t/n,i}$, $1 \leq i \leq n$ are independent and distributed like $X_{t/n}$. This shows that X_t is an infinitely divisible random variable. Let $\Psi(z)$ be the characteristic exponent of X_1 as defined in (2.1). Then it follows from property (i) that

$$\mathbb{E}[e^{izX_t}] = e^{-t\Psi(z)}.$$

In other words, X is completely determined by the triple (a, σ^2, Π) corresponding to the characteristic exponent of X_1. Accordingly, the function $\Psi(z)$ used in this context is called the ***characteristic exponent*** of X. We have seen that every Lévy process is naturally associated with an infinitely divisible random variable. It is also true, although more difficult to show, that every infinitely divisible random variable ξ gives rise to a (unique up to equality in distribution) Lévy process X such that $\xi \stackrel{d}{=} X_1$ (see Theorem 2.1 in [76]).

We gather now some conventions/notations with respect to the characteristic exponent of a Lévy process (these also apply to infinitely divisible random variables).

- The triple (a, σ^2, Π) is called the ***generating triple***. The quantity σ^2 is known as the ***Gaussian component*** and the measure $\Pi(\mathrm{d}x)$ is known as the ***Lévy measure***.

- The function $\mathbb{I}(|x| \leq 1)$ in (2.1) is known as a ***cut-off function***. Without additional restrictions on $\Pi(\mathrm{d}x)$, such a function is needed to ensure convergence of the integral. However, as is noted in [103] on pg. 38, any function $h(x)$ satisfying $h(x) = 1 + o(x)$ as $|x| \to 0$ and $h(x) = O(1/x)$ as $|x| \to \infty$ will suffice. We may therefore express (2.1) in various equivalent ways by varying $h(x)$ and a. If we choose a cut-off function $g(x)$ different from $\mathbb{I}(|x| \leq 1)$ we will write $(a, \sigma^2, \Pi)_{h \equiv g}$ for the generating triple. When we use the cut-off function $h(x) = \mathbb{I}(|x| \leq 1)$ we will simply use the notation (a, σ^2, Π) without subscript. If we need to be specific, we will refer to this as the ***canonical generating triple*** . Most commonly, we will choose $h(x) \equiv 0$ or $h(x) \equiv 1$ although in both cases we need more information about $\Pi(\mathrm{d}x)$ to justify this choice. The constant a, when used in the $(a, \sigma^2, \Pi)_{h \equiv 0}$ sense, is called the ***drift*** of the process.

- When $\Pi(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure with density function $\pi(x)$ we will call $\pi(x)$ a ***Lévy density***. When we refer to the generating triple in these cases we will write (a, σ^2, π) and analogously for other cut-off functions $h(x)$.

Often, we wish to work with the ***Laplace exponent*** of ξ. of a Lévy process X, which we may define via the characteristic exponent as

$$\begin{aligned} \psi(z) := \frac{1}{t} \log \mathbb{E}[e^{zX_t}] &= -\Psi(-iz) \\ &= \frac{\sigma^2 z^2}{2} + az + \int_{\mathbb{R}\setminus\{0\}} (e^{zx} - 1 - zx\mathbb{I}(|x| < 1))\Pi(\mathrm{d}x), \quad z \in i\mathbb{R}. \end{aligned} \tag{2.3}$$

Of course, this definition is rather useless unless we can extend $\psi(z)$ beyond just the imaginary numbers. From [76] we have the following equivalent condition to the existence of $\psi(z)$ in terms of the Lévy measure.

Theorem 1 (Theorem 3.6, in [76])**.** *Let $\psi(z)$ be the Laplace exponent of a Lévy process with generating triple (a, σ^2, Π). Then $\psi(z_0)$ is finite if, and only if $\int_{|x| \geq 1} e^{\mathrm{Re}(z_0)x} \Pi(\mathrm{d}x) < \infty$.*

For the majority of processes used explicitly in this work, except the family of stable process, we may always extend the domain of $\psi(z)$ to include a vertical strip of $\mathbb{C}$ containing the origin. In these cases we will usually work with the Laplace exponent instead of the characteristic exponent.

As is typical, we often classify Lévy processes by the characteristics of their sample paths. The first such classification deals with the amount of jump activity, which is measured by the number of discontinuities of a sample path over any finite interval. Each Lévy process has either almost surely finite jump activity or almost surely infinite jump activity. Therefore we can classify Lévy processes as either ***finite activity*** or ***infinite activity*** processes. Occasionally we may also simply write that the process *has* finite (resp. infinite) activity when referring to a finite (resp. infinite) activity process. We note that a Lévy process is a finite activity process if, and only if, the jumps follow a compound Poisson process with drift, i.e. if, and only if, $\Psi(z) = \frac{\sigma^2 z^2}{2} - iaz - \lambda \int_{\mathbb{R}} (e^{izx} - 1)\nu(\mathrm{d}x)$ for some $\lambda \in \mathbb{R}^+$, and probability measure $\nu(\mathrm{d}x)$ (see Section 2.6.1 in [76]).

We say a process is a ***finite variation*** (resp. ***infinite variation***) process if its sample paths have almost surely finite (resp. infinite) total variation. Occasionally we may also simply write that the process *has* finite (resp. infinite) variation when referring to a finite (resp. infinite) variation process. As with jump activity, each Lévy process has either finite variation, or infinite variation. Note that a process is a finite variation process if, and only if, $\int_{\mathbb{R}} \min(1, |x|)\Pi(\mathrm{d}x) < \infty$ and $\sigma = 0$ (see Section 2.6.1 in [76]).

The term ***subordinator*** refers to a Lévy process whose paths are almost surely increasing. A Lévy process which is not a subordinator but has no negative jumps is called a ***spectrally positive*** process. Likewise, a Lévy process which is not the negative of a subordinator, but has no positive jumps is called a ***spectrally negative*** process. Spectrally positive and negative processes are called ***spectrally one-sided processes***. Note that a scaled Brownian motion with drift is both a spectrally positive and a spectrally negative process. Lévy processes that have exclusively positive or exclusively negative jumps are called processes with ***one-sided*** jumps, or simply ***one-sided*** processes. When we want to emphasize that we are working with a process with both positive and negative jumps – i.e. the general setting – we will say that we are working with a process with ***two-sided*** jumps or simply a ***two-sided*** process. Finally, if X is a Lévy process then the process $-X$ is called the ***dual process*** or simply ***dual*** of X.

In some cases, we wish to consider a killed Lévy process. That is, we add an isolated point Δ to the state space $\mathbb{R}$ and assume that our Lévy process can reach Δ with positive probability. If it reaches Δ then the process remains there almost surely. Such a process is

called a ***killed Lévy process***. For a killed Lévy process X we will call the random variable

$$\zeta := \inf\{t \geq 0 : X_t = \Delta\}$$

the ***lifetime*** of X. Because a Lévy process is spatially homogeneous (see Definition 10.3 in [103]) the lifetime ζ must be exponentially distributed, that is, $\zeta \stackrel{d}{=} \mathbf{e}(q)$, where $\mathbf{e}(q)$ is an exponential random variable with parameter $q > 0$. We say that X is ***killed at rate*** q. If we follow the convention that when $q = 0$, $\mathbf{e}(q) = \infty$ almost surely, then we can describe any Lévy process in this way. We will often need this type of random variable in this work. Therefore, we close with the following statement:

> *The notation* $\boldsymbol{e}(q)$ *will always refer to an exponential random variable with parameter* $q > 0$. *In those cases where we need to consider* $q = 0$ *we assume* $\boldsymbol{e}(q) = \infty$ *a.s.*

2.1.4 Finance

All applications to finance in this thesis pertain to options on stock prices and all stock prices are modeled as the exponential of a Lévy process. We will always use A to denote the stock price process, where A is defined by $A_t := A_0 \exp(X_t)$ for some Lévy process X, and quantity $A_0 > 0$ representing the stock price at time 0. Further, we will always use $r > 0$ to denote the risk-free rate of return, and we assume that our probability measure $\mathbb{P}$ is risk neutral. In other words, under $\mathbb{P}$, the discounted stock price process, defined by $\exp(-rt)A_t$, is a martingale. For a Lévy process X the reader may verify this latter condition holds if, and only if, $\psi(1) = r$ where $\psi(z)$ is the Laplace exponent of X.

For a Lévy process X and a point $x \in \mathbb{R}$ we define the ***first passage time*** τ_x^+ (resp. τ_x^-) above (resp. below) x by

$$\tau_x^+ := \inf\{t \geq 0 : X_t > x\} \qquad \left(\text{resp. } \tau_x^- := \inf\{t \geq 0 : X_t < x\}\right).$$

The associated random variables $X_{\tau_x^+} - x$ and and $x - X_{\tau_x^+ -}$ are called the ***overshoot*** and ***undershoot*** for the level x. Finally, for a real number z we define $z^+ := \max\{z, 0\}$.

2.2 Families of Lévy processes

We summarize here the families of Lévy processes which appear in this dissertation. In particular we will list Lévy measures, and characteristic/Laplace exponents, but will largely forgo their derivations as these are well documented in the references we provide.

2.2.1 Stable processes

A well-known property of a Brownian motion B is that it satisfies the property $t^{1/2}B_1 \stackrel{d}{=} B_t$ for all $t \geq 0$. That is, we can achieve a scaling in time by instead scaling in space and vice versa. This property, known as ***self-similarity***, is not unique to Brownian motion. In fact there are many other examples of self-similar processes, among these the class of stable processes. A random variable ξ is called a ***stable random variable*** [1] if it satisfies the property

$$c_n \xi \stackrel{d}{=} \xi_1 + \ldots + \xi_n, \quad t \geq 0,\, n \in \mathbb{N},$$

where the $\{\xi_i\}$ are i.i.d random variables, equal in distribution to ξ. It can be shown, (see Section VI.1 in [43]) that c_n is a simple function of n, namely $c_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$. Evidently, stable random variables are infinitely divisible, and from Theorem C.1 in [116] we know that their Lévy measures are absolutely continuous with respect to the Lebesgue measure. In particular, they have a Lévy density given by

$$\pi(x) = \mathbb{I}(x < 0)\frac{c_1}{|x|^{1+\alpha}} + \mathbb{I}(x > 0)\frac{c_2}{x^{1+\alpha}}, \tag{2.4}$$

where $c_1,\, c_2 \geq 0$. A ***stable process*** is just a Lévy process defined by the generating triple of a stable random variable. From (2.4), we see that stable processes are necessarily infinite activity processes, with paths of infinite variation only when $\alpha \geq 1$. One may show (see for example exercise 1.4 in [76]) that for $\alpha \in (0, 1) \cup (1, 2)$ the density $\pi(x)$ yields a characteristic exponent of the form

$$\Psi(z) = c|z|^{\alpha}\left(1 - i\beta \tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}(z)\right), \tag{2.5}$$

where $c = d_1 + d_2$, $\beta = (d_2 - d_1)/(d_1 + d_2)$, and $d_i = -c_i\Gamma(-\alpha)\cos(\pi\alpha/2)$, for $i \in \{1, 2\}$. Here, we should interpret $\Psi(z)$ in the $(0, 0, \pi)_{h\equiv 0}$ sense for $\alpha \in (0, 1)$ and in the $(0, 0, \pi)_{h\equiv 1}$ sense for $\alpha \in (0, 2)$. If we abide by the convention that $0 \times \infty = 0$ then one can show that (2.5)

[1] Technically, we are speaking about *strictly* stable random variables. Adhering to the original definition, stable random variables satisfy the property $\xi_1 + \ldots + \xi_n = c_n\xi + b_n$ for any $n \in \mathbb{N}$ and constants c_n and b_n.

also holds for parameters $(\alpha, \beta) = (1, 0)$, where c is now defined by $c = \pi c_1$ (the number π, not the density) and we interpret $\Psi(z)$ in the $(0, 0, \pi)$ sense. From (2.5) it is easy to deduce the self-similarity property for stable processes with our chosen parameters. That is, for a stable process X with a characteristic exponent of the form (2.5), the following holds:

$$\{\lambda^{1/\alpha} X_t : t \geq 0\} \stackrel{d}{=} \{X_{\lambda t} : t \geq 0\}, \quad \lambda > 0. \tag{2.6}$$

For our purposes in Chapters 3 and 7 it will be useful to consider a reparametrization of the process. The standard approach (see for example [68] and [70]) is to notice from (2.5) that c is just a scaling parameter, and in particular, we may assume the following normalization without loss of generality:

$$c = \left(1 + \beta^2 \tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{-\frac{1}{2}}.$$

Now, instead of the parameter β we use the positivity parameter ρ defined in [115] as

$$\rho = \mathbb{P}(X_1 > 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1}\left(\beta \tan\left(\frac{\pi\alpha}{2}\right)\right).$$

Using standard trigonometric identities we may then re-write characteristic exponent of the normalized process in terms of the parameter ρ as follows:

$$\Psi(z) = \mathbb{I}(z < 0) e^{-\pi i \alpha (1-2\rho)/2} |z|^\alpha + \mathbb{I}(z > 0) e^{\pi i \alpha (1-2\rho)/2} |z|^\alpha. \tag{2.7}$$

A few comments about the processes resulting from other choices of the parameters are in order. When $\alpha = 1$ and $\beta \neq 0$ we obtain processes for which the self-similarity property fails. This makes their analysis significantly different from self-similar processes, and there will be no references to non-self-similar stable processes in this work. When $\alpha \in (0, 1)$ and $\beta = \pm 1$ the resulting process is a subordinator/the negative of a subordinator. As these cases are not interesting for our purposes, they will also be omitted here. Finally, the case $\alpha = 2$ yields the family of scaled Brownian motions which has been well studied. We will refer to scaled Brownian motions in this dissertation, but not in the context of stable processes. Therefore, references to stable processes in this work will be limited to the following set of admissible parameters:

$$\mathcal{P} = \{\alpha \in (0, 1), \beta \in (-1, 1)\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha \in (1, 2), \beta \in [-1, 1]\}.$$

One may show that this can be equivalently expressed in terms of the parameters (α, ρ). That is $(\alpha, \beta) \in \mathcal{P}$ if, and only if, $(\alpha, \rho) \in \mathcal{A}$, where

$$\mathcal{A} = \{\alpha \in (0,1), \rho \in (0,1)\} \cup \{\alpha = 1, \rho = \tfrac{1}{2}\} \cup \{\alpha \in (1,2), \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}. \tag{2.8}$$

The self-similarity property ensures that there are many applications of stable processes in the natural sciences, where they often go by the name "Lévy flights" (see [41]). However, their use in the context of mathematical finance, for example to model log-stock prices, is hampered by the fact that stable random variables are heavy-tailed. In particular, other than the special case $\alpha = 2$, no stable process has finite exponential moments. As a consequence, an exponentiated stable process cannot be used to represent a stock price since financial theory dictates that discounted stock prices must be martingales. For a discussion of the use of stable processes in finance see Chapter 7 in [33] and Sections 5 and 6 in [52].

2.2.2 Processes with jumps of rational transform

The simplest way to incorporate jumps into the standard geometric Brownian Motion model for asset prices is to consider the addition of compound Poisson jumps. This formula has been used in mathematical finance since 1976 when Merton [86] proposed a jump-diffusion model whose jumps were governed by a compound Poisson process with normal jumps. More recently, Kou [63] proposed replacing the normal jumps with jumps having an asymmetric double exponential distribution, which yielded a more analytically tractable model for pricing stock options and also provided a better fit to observed stock price data. What makes this model tractable is that the Laplace transform of a double exponential distribution is a rational function, and thus the Laplace exponent of a process with jumps distributed in this way is also necessarily a rational function. Naturally, we wish to generalize this concept, which leads to the following definition. Lévy processes with ***jumps of rational transform*** have Lévy densities of the form

$$\pi(x) = \lambda \left(\mathbb{I}(x<0) \sum_{n=1}^{\hat{N}} \sum_{j=1}^{\hat{M}_n} \hat{c}_{nj} (\hat{\rho}_n)^j \frac{|x|^{j-1}}{(j-1)!} e^{\hat{\rho}_n x} + \mathbb{I}(x>0) \sum_{n=1}^{N} \sum_{j=1}^{M_n} c_{nj} (\rho_n)^j \frac{x^{j-1}}{(j-1)!} e^{-\rho_n x} \right),$$

where $\lambda > 0$ is a parameter determining the frequency of the jumps, and $\tilde{\pi}(x) = \lambda^{-1}\pi(x)$ is the density of a probability distribution. The remaining parameters have the following properties: $\hat{\rho}_1, \rho_1 > 0$, $\mathrm{Re}(\hat{\rho}_n) > 0$, $2 \le n \le \hat{N}$, and $\mathrm{Re}(\rho_n) > 0$, $2 \le n \le N$, and the other parameters may be complex numbers. The indexing on the parameters $\{\hat{\rho}_n\}_{1 \le n \le \hat{N}}$ and

$\{\rho_n\}_{1\leq n\leq N}$ is assigned such that they are ordered by their real component, i.e.

$$\rho_1 < \mathrm{Re}(\rho_2) \leq \ldots \leq \mathrm{Re}(\rho_N),$$
$$\hat{\rho}_1 < \mathrm{Re}(\hat{\rho}_2) \leq \ldots \leq \mathrm{Re}(\hat{\rho}_N).$$

Clearly, any continuous random variable with rational Laplace transform must have a density of the form $\tilde{\pi}(x)$. By direct application of (2.3) with cut-off function $h(x) = 0$, we see that a process with jump distribution determined by $\tilde{\pi}(x)$ will have Laplace exponent of the form

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + \lambda \left(\sum_{n=1}^{\hat{N}} \sum_{j=1}^{\hat{M}_n} \hat{c}_{nj} \left(\frac{\hat{\rho}_n}{\hat{\rho}_n + z} \right)^j + \sum_{n=1}^{N} \sum_{j=1}^{M_n} c_{kj} \left(\frac{\rho_n}{\rho_n - z} \right)^j - 1 \right), \qquad (2.9)$$
$$\mathrm{Re}(z) \in (-\hat{\rho}_1, \rho_1).$$

It is clear that we may continue $\psi(z)$ meromorphically to a rational function on $\mathbb{C}$.

Next, we gather some important facts about $\psi(z)$, mainly concerning the zeros of the function $q - \psi(z)$ where $q > 0$. The importance of the zeros will become clear in our discussion of the Wiener-Hopf factorization in Chapter 3 and the exponential functional in Chapter 4. The proof of these facts, which are gathered in Theorem 2, appears in [46, 83] and [69]. First, we note that $\psi(z)$ has N (resp. $\hat{N}$) poles at the points $\{\rho_n\}_{1\leq n\leq N}$ (resp. $\{-\hat{\rho}_n\}_{1\leq n\leq \hat{N}}$) which are all located in the half-plane $\mathrm{Re}(z) > 0$ (resp. $\mathrm{Re}(z) < 0$) with corresponding multiplicities $\{M_n\}_{1\leq n\leq N}$ (resp. $\{\hat{M}_n\}_{1\leq n\leq \hat{N}}$). Now we define the following quantities:

- U denotes the degree of the numerator of $\psi(z)$.
- $P = \sum_{1\leq n\leq N} M_n$ (resp. $\hat{P} = \sum_{1\leq n\leq \hat{N}} \hat{M}_n$) denotes the pole count (with multiplicity) in the half-plane $\mathrm{Re}(z) > 0$ (resp. $\mathrm{Re}(z) < 0$).
- K (resp. $\hat{K}$) denotes the zero count (with multiplicity) of the function $q - \psi(z)$ in the half-plane $\mathrm{Re}(z) > 0$ (resp. $\mathrm{Re}(z) < 0$).
- $\{\zeta_n\}_{1\leq n\leq K}$ and $\{-\hat{\zeta}_n\}_{1\leq n\leq \hat{K}}$ denote the zeros of $q - \psi(z)$ where the ζ_n (resp. $\hat{\zeta}_n$) are ordered according to their real part, e.g. $\mathrm{Re}(\zeta_n) \leq \mathrm{Re}(\zeta_{n+1})$.

Theorem 2. *Let X be a Lévy process with jumps of rational transform, and $q > 0$.*

1. *$K + \hat{K}$ gives a complete zero count (with multiplicity) of the function $q - \psi(z)$ and $U = K + \hat{K}$.*

2. *ζ_1 and $\hat{\zeta}_1$ are real positive numbers. Moreover, for all $n \neq 1$, $\zeta_1 < \mathrm{Re}(\zeta_n)$ and $\hat{\zeta}_1 < \mathrm{Re}(\hat{\zeta}_n)$.*

3. *If $\sigma > 0$ then $K = P + 1$ and $\hat{K} = \hat{P} + 1$.*

4. *If $\sigma = 0$, and $a > 0$ (resp. $a < 0$) then $K = P + 1$ and $\hat{K} = \hat{P}$ (resp. $\hat{K} = \hat{P} + 1$ and $K = P$).*

5. *If $\sigma = a = 0$, then $K = P$ and $\hat{K} = \hat{P}$.*

6. *There exist at most $2U - 1$ complex numbers q such that equation $\psi(z) = q$ has solutions of multiplicity greater than one.*

7. *As $q \to 0^+$ we have the following possibilities:*

$$\begin{cases} \text{if } \mathbb{E}[X_1] > 0, \text{ then } \zeta_1(0^+) = 0 \text{ and } \hat{\zeta}_1(0^+) > 0, \\ \text{if } \mathbb{E}[X_1] < 0, \text{ then } \zeta_1(0^+) > 0 \text{ and } \hat{\zeta}_1(0^+) = 0, \\ \text{if } \mathbb{E}[X_1] = 0, \text{ then } \zeta_1(0^+) = 0 \text{ and } \hat{\zeta}_1(0^+) = 0. \end{cases}$$

An important subclass of processes with jumps of rational transform is the class of hyper-exponential processes. We briefly describe these processes as they are exceptionally useful for performing calculations. Also, they are the basis for examples and discussion in Chapters 3, 4, 5, 6, and 8 and are the finite dimensional equivalent of the class of meromorphic Lévy processes which are discussed in the next section. A Lévy process with ***hyper-exponential jumps*** or simply a ***hyper-exponential*** Lévy process is a Lévy process with Lévy density

$$\pi(x) = \lambda \left(\mathbb{I}(x < 0) \sum_{n=1}^{\hat{N}} \hat{c}_n \hat{\rho}_n e^{\hat{\rho}_n x} + \mathbb{I}(x > 0) \sum_{n=1}^{N} c_n \rho_n e^{-\rho_n x} \right), \tag{2.10}$$

where all the coefficients are positive and $\sum_{1 \leq n \leq N} c_n + \sum_{1 \leq n \leq \hat{N}} \hat{c}_n = 1$. In particular, $\tilde{\pi}(x) = \lambda^{-1}\pi(x)$ is the density of a finite mixture of exponential distributions. Using the fact that $\sum_{1 \leq n \leq N} c_n + \sum_{1 \leq n \leq \hat{N}} \hat{c}_n = 1$ and re-labeling coefficients $a_n := \lambda c_n$, $\hat{a}_n := \lambda \hat{c}_n$ we see from (2.9) that the Laplace exponent of a hyper-exponential process is given by,

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z \sum_{n=1}^{N} \frac{a_n}{\rho_n - z} - z \sum_{n=1}^{\hat{N}} \frac{\hat{a}_n}{\hat{\rho}_n + z}, \quad \mathrm{Re}(z) \in (-\hat{\rho}_1, \rho_1). \tag{2.11}$$

This is the $(a, \sigma^2, \Pi)_{h\equiv 0}$ representation of $\psi(z)$. For comparison with meromorphic processes (next section) we also give the $(a, \sigma^2, \Pi)_{h\equiv 1}$ representation

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z^2 \sum_{n=1}^{\hat{N}} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n - z)}, \quad \mathrm{Re}(z) \in (-\hat{\rho}_1, \rho_1), \tag{2.12}$$

which we may verify by direct calculation by using (2.10), (2.3), and cut-off function $h(x) = 1$. We see that, whichever representation we choose, $\psi(z)$ can be meromorphically continued to a rational function on $\mathbb{C}$ with simple poles at points $\{-\hat{\rho}_n\}_{1\leq n\leq \hat{N}}$, and $\{\rho_n\}_{1\leq n\leq N}$. Again, we are interested in the solutions of the equation $\psi(z) = q$, and in this case we are able to say more than the results of Theorem 2. It is quite easy to see from (2.11), essentially by invoking the intermediate value theorem (see also Lemma 2.1 in [25]), that the solutions are all real and have the following important interlacing property

$$\begin{aligned} &0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 \dots \\ &0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 \dots . \end{aligned} \tag{2.13}$$

2.2.3 Meromorphic processes

The benefit of working with processes like hyper-exponential processes, is that they are very tractable and suitable for calculation. The main drawback is that they necessarily have finite activity. This presents a problem, especially for modeling financial assets, since empirical evidence (see for example [3, 29]) suggests that infinite activity processes are the most suitable for modeling share prices. The class of meromorphic Lévy processes recently introduced in a series of papers [65, 66, 71] strikes a balance between tractability and fit. Roughly, we may think of meromorphic Lévy processes as the "infinite version" of hyper-exponential processes. This derives from the fact that we obtain a meromorphic Lévy process essentially by replacing the finite sums in the Lévy measure of a hyper-exponential process (2.10) by infinite series (the reader should compare (2.10) and (2.14) as well as (2.12) and (2.15)). In this way we may build infinite activity and infinite variation processes, while still maintaining a tractable model.

We define a ***meromorphic process*** to be a Lévy process whose Lévy density has the form

$$\pi(x) = \mathbb{I}(x < 0) \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x} + \mathbb{I}(x > 0) \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x}, \tag{2.14}$$

where the coefficients are positive and the sequences $\{\rho_n\}_{n\geq 1}$ and $\{\hat{\rho}_n\}_{n\geq 1}$ are strictly increasing. Further we stipulate that the series $\sum_{n\geq 1} a_n\rho_n^{-2}$, and $\sum_{n\geq 1}\hat{a}_n\hat{\rho}_n^{-2}$ both converge as this is equivalent to the integrability condition (2.2) on $\pi(x)$. The name "meromorphic" references the fact that a process in the meromorphic class has a Laplace exponent $\psi(z)$ which is a real meromorphic function. We recall that a function $g:\mathbb{C}\mapsto\mathbb{C}\cup\{\infty\}$ is called a ***meromorphic function*** if it has no singularities other than poles. It is called a ***real meromorphic function*** if, in addition to being meromorphic, it satisfies $g(z)\in\mathbb{R}\cup\{\infty\}$ for all $z\in\mathbb{R}$, or, equivalently if $\overline{g(z)}=g(\overline{z})$. By direct calculation using (2.14) and (2.3), we may verify that the $(a,\sigma^2,\Pi)_{h\equiv 1}$ representation of the Laplace exponent is

$$\psi(z)=\frac{\sigma^2z^2}{2}+az+z^2\sum_{n\geq 1}\frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n+z)}+z^2\sum_{n\geq 1}\frac{a_n}{\rho_n(\rho_n-z)},\quad \mathrm{Re}(z)\in(-\hat{\rho}_1,\rho_1). \tag{2.15}$$

The function $\psi(x)$ may be continued to a meromorphic function on $\mathbb{C}$ with simple poles at points $\{-\hat{\rho_n}\}_{n\geq 1}$, $\{\rho_n\}_{n\geq 1}$, and the equation $\psi(z)=q$ has solutions $\{-\hat{\zeta}_n\}_{n\geq 1}$ and $\{\zeta_n\}_{n\geq 1}$ which are real numbers and satisfy the familiar interlacing property

$$\begin{aligned}&0<\zeta_1<\rho_1<\zeta_2<\rho_2<\dots\\&0<\hat{\zeta}_1<\hat{\rho}_1<\hat{\zeta}_2<\hat{\rho}_2<\dots.\end{aligned} \tag{2.16}$$

A non-trivial and important fact proven in [66] is that these are, in fact, the only solutions of the equation $\psi(z)=q$.

Two important classes of meromorphic processes are the β and θ-class introduced in [65] and [66] respectively. A process in the β-***class*** or simply a β-***process*** is a meromorphic process with a Lévy density of the form

$$\pi(x)=\mathbb{I}(x<0)c_1\frac{e^{\alpha_1\beta_1x}}{(1-e^{\beta_1x})^{\lambda_1}}+\mathbb{I}(x>0)c_2\frac{e^{-\alpha_2\beta_2x}}{(1-e^{-\beta_2x})^{\lambda_2}}, \tag{2.17}$$

where c_i, α_i, $\beta_i>0$ and $\lambda_i\in(0,3)/\{1,2\}$. With the help of the binomial series, one may verify that $\pi(x)$ has the form (2.14) and specifically, that

$$\begin{gathered}\hat{\rho}_n=\beta_1(\alpha_1+n-1),\quad \rho_n=\beta_2(\alpha_2+n-1),\\ \hat{a}_n=\hat{\rho}_n^{-1}c_1\binom{n+\lambda_1-2}{n-1},\quad \text{and}\quad a_n=\rho_n^{-1}c_2\binom{n+\lambda_2-2}{n-1}.\end{gathered}$$

Choosing $\lambda_i>1$ yields an infinite activity processes, and if $\lambda_i>2$ then the process will also have infinite variation. If $\sigma=0$ and $\lambda_1,\lambda_2<1$ the result is a finite activity process. Thus,

the β-class is flexible enough to yield all manner of jump behaviour. The origin of its name is its Laplace exponent, which has the following closed-form expression,

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \frac{c_1}{\beta_1}\left(\mathrm{B}\left(\alpha_1 + \frac{z}{\beta_1}, 1-\lambda_1\right) - \mathrm{B}(\alpha_1, 1-\lambda_1)\right) + \frac{c_2}{\beta_2}\left(\mathrm{B}\left(\alpha_2 - \frac{z}{\beta_2}, 1-\lambda_2\right) - \mathrm{B}(\alpha_2, 1-\lambda_2)\right),$$

where $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function, and μ is determined in terms of a by the condition $\psi'(0) = a$.

A process in the θ-***class*** or simply a θ-***process*** is a meromorphic process with a Lévy density of the form

$$\pi(x) = \mathbb{I}(x<0)c_1\beta_1 e^{\alpha_1 x}\Theta_k(-x\beta_1) + \mathbb{I}(x>0)c_2\beta_2 e^{-\alpha_2 x}\Theta_k(x\beta_2), \tag{2.18}$$

where all the parameters other than k are positive. The function $\Theta_k(x)$ is the kth order (fractional) derivative of the theta function $\theta_3(e^{-x})$, that is,

$$\Theta_k(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\theta_3(e^{-x}) = \delta_{k,0} + 2\sum_{n\geq 1} n^{2k} e^{-n^2 x}, \quad x>0. \tag{2.19}$$

The activity of the jumps and variation of the paths are controlled by the parameter $\chi = k + 1/2$ where $\chi \in (0,3)$. When $\chi < 1$ and $\sigma = 0$ the process is a compound Poisson process, when $\chi > 1$ we obtain an infinite activity process, and when $\chi > 2$ we obtain an infinite variation process. From (2.18) and (2.19) we may verify that (2.18) has the form (2.14) with

$$\hat{\rho}_n = \alpha_1 + n^2\beta_1, \quad \rho_n = \alpha_2 + n^2\beta_2, \quad \hat{a}_n = 2\hat{\rho}_n^{-1}c_1\beta_1 n^{2\chi-1}, \quad \text{and} \quad a_n = 2\rho_n^{-1}c_2\beta_2 n^{2\chi-1}.$$

As with β-processes we may obtain a closed form expression for the Laplace exponent, but only for certain values of χ. For example, when $\chi = (2j+1)/2$, $j \in \{1, 2\}$ we obtain the Laplace exponent,

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \gamma + (-1)^j\left(c_1\pi\left(\sqrt{(\alpha_1+z)/\beta_1}\right)^{2j-1}\coth\left(\pi\sqrt{(\alpha_1+z)/\beta_1}\right) + c_2\pi\left(\sqrt{(\alpha_2-z)/\beta_2}\right)^{2j-1}\coth\left(\pi\sqrt{(\alpha_2-z)/\beta_2}\right)\right),$$

where the constant γ is determined by the condition $\psi(0) = 0$. The constant μ is equal to $a - \int_{\mathbb{R}\setminus\{0\}} x\pi(x)\mathrm{d}x$ when $\chi = 3/2$, and is determined in terms of a by the condition $\psi'(0) = a$ for $\chi = 5/2$.

2.2.4 Completely monotone processes

The class of completely monotone processes is large and important. In fact, it includes all of the processes discussed thus far, except for those processes with jumps of rational transform which are not hyper-exponential processes. It also includes the most popular processes in mathematical finance.

We recall that a ***completely monotone function*** is an infinitely differentiable function, $f : \mathbb{R}^+ \to \mathbb{R}$, that obeys the property:

$$(-1)^n f^{(n)}(x) \geq 0 \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } x > 0.$$

By Bernstein's theorem we have an equivalent way to define $f(x)$, namely, $f(x)$ is a completely monotone function if, and only if, there exists a measure μ concentrated on $\bar{\mathbb{R}}^+$ such that

$$f(x) = \int_{\bar{\mathbb{R}}^+} e^{-xu}\mu(\mathrm{d}u)$$

for all $x > 0$. Accordingly, we define a Lévy process with ***completely monotone jumps*** or simply a ***completely monotone*** Lévy process as a process whose Lévy measure has a density of the form

$$\pi(x) = \mathbb{I}(x < 0)\int_{\mathbb{R}^-} e^{-ux}\mu(\mathrm{d}u) + \mathbb{I}(x > 0)\int_{\mathbb{R}^+} e^{-ux}\mu(\mathrm{d}u), \tag{2.20}$$

where $\mu(\mathrm{d}u)$ is a measure concentrated on $\mathbb{R}\setminus\{0\}$, that satisfies the condition

$$\int_{\mathbb{R}\setminus\{0\}} \frac{1}{|u|(1+u^2)}\mu(\mathrm{d}u) < \infty. \tag{2.21}$$

One may verify that condition (2.21) on $\mu(\mathrm{d}u)$ is equivalent to the integrability condition (2.2) on $\pi(x)$. We will call $\mu(\mathrm{d}u)$ the ***representing measure***.

As an example, let us consider a hyper-exponential process as a completely monotone pro-

cesses. If we let

$$\mu(\mathrm{d}x) = \sum_{n=1}^{\hat{N}} \lambda \hat{c}_n \hat{\rho}_n \delta_{-\hat{\rho}_n}(\mathrm{d}x) + \sum_{n=1}^{N} \lambda c_n \rho_n \delta_{\rho_n}(\mathrm{d}x), \tag{2.22}$$

then we may easily recover the density of the Lévy measure of a hyper-exponential process (2.10) via $\mu(\mathrm{d}u)$ and (2.20). In fact, an equivalent way to define a hyper-exponential process is to define it as a completely monotone process whose representing measure $\mu(\mathrm{d}u)$ has finite support in $\mathbb{R}$. We may define a meromorphic process in an analogous way, namely, a meromorphic process is a completely monotone process whose representing measure is supported on a subset of $\mathbb{R}$ that is infinite, countable, and without accumulation points.

Completely monotone processes appear frequently in mathematical finance. Beside the models already mentioned, popular models like normal inverse Gaussian (NIG) processes, variance gamma (VG) processes, and generalized tempered stable processes are all examples of completely monotone processes. We briefly introduce these three families here. Our description is based on the concept that all three derive in some fashion from tempered stable subordinators.

A ***tempered stable subordinator*** is a Lévy process with Lévy measure

$$\Pi(\mathrm{d}x) = \frac{ce^{-\rho x}}{x^{1+\alpha}}\mathrm{d}x, \tag{2.23}$$

where c, $\rho > 0$, $\alpha \in (0,1)$, and $\Pi(\mathrm{d}x)$ is concentrated on $\mathbb{R}^+$. We see immediately that the name "tempered stable" originates from the fact that this process is derived by tempering the Lévy measure of a stable subordinator with an exponential term. This has the effect of allowing the small jumps to retain the behaviour of a stable process while simultaneously reducing the likelihood of extremely large jumps.

In order to continue the discussion, we require a technique known as Brownian subordination. Consider a subordinator S and an independent scaled Brownian motion with drift B given by $B_t := at + \sigma W_t$, where W is a standard Brownian motion. We will denote the Laplace exponents of B and S by $\psi_B(z)$ and $\psi_S(z)$ respectively. The process X given by $X_t := B_{S_t} = aS_t + \sigma W_{S_t}$ is a ***subordinated Brownian motion***, and the procedure of time changing a scaled Brownian motion with drift in this way is called ***(Brownian) subordination (of B by S)***. We see that the effect of subordination is to change the time scale of B to a random (but strictly increasing) time. Interestingly, the resulting process X

is a Lévy process with Laplace exponent $\psi(z) = \psi_S(\psi_B(z))$ (see Lemma 2.15 in [76]). In fact, the previous statements are true if we apply subordination to any Lévy process (not just scaled Brownian motions) although for our purposes Brownian subordination will suffice.

Now, setting $\alpha = 1/2$ we obtain from (2.23) an inverse Gaussian subordinator (see pg. 9 in [76] for more details). The infinite variation process that results from Brownian subordination by this subordinator is called a ***NIG*** process. Its small jumps behave like those of a stable process with parameter $\alpha = 1$. Since we will not be using NIG processes extensively in this work, and since the expression of the Lévy measure of a NIG process is fairly complicated, we will omit stating it, and the expression of the Laplace exponent, here. For further details of the NIG process we refer the interested reader to table 4.5 in [33].

To obtain a VG process we extend the range of α in the definition of the Lévy measure of the tempered stable subordinator (2.23), to include the case $\alpha = 0$. Such a process is called a ***gamma subordinator*** since for each $t > 0$ it has a transition density which is equal to the density of a gamma distribution with parameters ct, and ρ. By Brownian subordination using a gamma subordinator we obtain a VG process. Alternatively, and perhaps more directly, one may obtain a VG process simply as the difference of two independent gamma subordinators. Specifically, a ***VG*** process is a Lévy process with Lévy density

$$\pi(x) = \mathbb{I}(x < 0)\frac{ce^{\hat{\rho}x}}{|x|} + \mathbb{I}(x > 0)\frac{ce^{-\rho x}}{x}, \tag{2.24}$$

where c, $\hat{\rho}$, $\rho > 0$. In particular, we see from the density that the VG process is an infinite activity, finite variation process. From (2.24) and (2.3) we may derive the $(a, \sigma^2, \Pi)_{h\equiv 0}$ representation of the Laplace exponent, which has the form

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az - c\log\left(1 + \frac{z}{\hat{\rho}}\right) - c\log\left(1 - \frac{z}{\rho}\right), \quad \mathrm{Re}(z) \in (-\hat{\rho}, \rho). \tag{2.25}$$

Via inversion of the Laplace transform in (2.24), we may also identify the representing measure:

$$\mu(\mathrm{d}u) = \mathbb{I}(u < -\hat{\rho})c\mathrm{d}u + \mathbb{I}(u > \rho)c\mathrm{d}u.$$

Finally, to obtain a generalized tempered stable process we extend once more the range of α in the definition of the Lévy measure of the tempered stable subordinator to include the cases $\alpha < 2$. A generalized tempered stable process is defined as the difference of two independent

processes with such Lévy densities. That is, a ***generalized tempered stable*** process is a Lévy process with a Lévy density of the form

$$\pi(x) = \mathbb{I}(x<0)\frac{\hat{c}}{|x|^{1+\hat{\alpha}}}e^{\hat{\rho}x} + \mathbb{I}(x>0)\frac{c}{x^{1+\alpha}}e^{-\rho x}, \tag{2.26}$$

where $\hat{c}$, $\hat{\rho}$, c, $\rho > 0$ and $\hat{\alpha}, \alpha < 2$. This structure is rich enough to support all types of jump behaviour. Setting $\hat{\alpha}, \alpha < 0$ gives a finite activity process, whereas an infinite activity process results from setting $\hat{\alpha}, \alpha \geq 0$. The choice $\hat{\alpha}, \alpha \geq 1$ results in an infinite variation process. From (2.26) and (2.3) and for α, $\hat{\alpha} \in (-1,2)\backslash\{0,1\}$ we may obtain (see Proposition 4.2 in [33]) the $(a, \sigma^2, \Pi)_{h\equiv 1}$ representation of the Laplace exponent. This has the form

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \Gamma(-\hat{\alpha})\hat{c}\big((\hat{\rho}+z)^{\hat{\alpha}} - \hat{\rho}^{\hat{\alpha}}\big) + \Gamma(-\alpha)c\big((\rho - z)^{\alpha} - \rho^{\alpha}\big), \tag{2.27}$$
$$\mathrm{Re}(z) \in (-\hat{\rho}, \rho),$$

where μ is determined in terms of a by the condition $\psi'(0) = a$. The Laplace exponents for the cases corresponding to the remaining choices of α and $\hat{\alpha}$ may also be obtained in closed form, but we omit stating these here as we will not make use of them in this thesis. The interested reader should consult Proposition 4.2 in [33]. However, we notice that the case $\hat{\alpha}$, $\alpha = 0$, $\hat{c} = c$ corresponds to the VG process, whose Laplace exponent we have provided in (2.25). The process defined by the parameter choice $\hat{c} = c$, $\hat{\alpha} = \alpha \in (-1,2)\backslash\{0,1\}$ corresponds to the well-known CGMY process (see [29]). The name CGMY derives from the initials of the authors of [29], and usually in the CGMY representation of a generalized tempered stable process we will write $\hat{c} = c = C$, $\hat{\rho} = G$, $\rho = M$, and $\hat{\alpha} = \alpha = Y$.

We will limit our use of generalized tempered stable processes to those processes for which $\hat{\alpha} = \alpha \in (-1,2)$. There are two reasons for this: first, generalized tempered stable processes are not completely monotone when $\hat{\alpha}$, $\alpha < -1$; and second, generalized tempered stable processes can be obtained by Brownian subordination if, and only if, $\alpha = \hat{\alpha} \in (-1,2)$ and $\hat{c} = c$ (see Proposition 4.1 in [33]). When $\hat{\alpha}$, $\alpha \in (-1,2)$ we may identify the representing measure via Laplace transform inversion in (2.26) as,

$$\mu(\mathrm{d}u) = \mathbb{I}(u<-\hat{\rho})\hat{c}\frac{|u+\hat{\rho}|^{\hat{\alpha}}}{\Gamma(1+\hat{\alpha})}\mathrm{d}u + \mathbb{I}(u>\rho)c\frac{(u-\rho)^{\alpha}}{\Gamma(1+\alpha)}\mathrm{d}u.$$

Generalized tempered stable processes, and especially VG processes, have become very popular in mathematical finance for a number of reasons. First, there are some proponents, for example the authors of [3,56], of the theory that the classical Black Scholes framework, where

asset prices are modeled by geometric Brownian motion, should be abandoned in favour of a model where log-asset prices are pure jump Lévy processes with infinite activity and finite variation. Of these, the VG process lends itself well to simulation and (certain) calculations as it has a known transition density and satisfies the desired activity/path properties (when $\sigma = 0$). Additionally, when log asset prices follow a VG process, the distribution of the log-returns has exponentially decaying tails, i.e. heavier than the normal distribution but not as heavy as those of a stable distribution, which is a desirable property.

Chapter 3

The Wiener-Hopf factorization

This chapter gives a brief introduction to the Wiener-Hopf Factorization for Lévy processes. We begin by stating a few necessary definitions and then we describe the fundamental theorems and identities which are collectively known as the Wiener-Hopf factorization. Although a proof of these is well beyond the scope of this work we give a brief description of the intuition behind the proofs that are based on path decompositions, e.g. those found in [14], [50], and [76]. Next we give specific results for families of Lévy processes for which the Wiener-Hopf factors are known explicitly. This is intended as a summary of the research activity in this field of approximately the last ten years. We also include a discussion on the techniques which are used to derive the factorizations employing the hyper-exponential process as an example. Finally we give two examples of direct applications of the Wiener-Hopf factorization in finance, specifically in the pricing of barrier options and of perpetual American options.

3.1 Introduction

In this work the Wiener-Hopf factorization is an important tool which we can use to solve option pricing problems. This is, however, a very narrow use for a versatile and important object from the theory of Lévy processes. From a theoretical point of view, the Wiener-Hopf factorization is the cornerstone of fluctuation theory. From a practical point of view, its applications extend well beyond mathematical finance, for example to risk management, actuarial science, and any other field where there is a need to solve exit problems.

In this Introduction we state the collection of theorems and identities known as the Wiener-Hopf factorization in three parts. In Theorem 3, Part 1, we give the key identities. We then

present an informal discussion on the derivation of the identities for random walks followed by an explanation of the analogous objects for Lévy processes. The discussion is meant to familiarize the reader with the intuition behind the identities of Part 1 without delving into the intricate details of the full proof for Lévy processes. During the course of this discussion we introduce ladder processes and local time; these concepts are necessary to state Theorem 3, Part 2. In Theorem 3, Part 3 we consider the analytical properties of the Wiener-Hopf factors and, importantly, how we can determine the Wiener-Hopf factors from these properties.

There are two remaining sections in this chapter. Section 3.2 discusses three important methods for determining the Wiener-Hopf factors. The methods are all demonstrated using the example of the hyper-exponential process. The remainder of Section 3.2 is devoted to a thorough review of the processes for which we have explicit expressions for the Wiener-Hopf factors. The expressions themselves are presented, and when possible, we give a brief description of the method of proof. In Section 3.3 we give two important examples of an application of the Wiener-Hopf factorization in finance. The first of these examples provides a preview for the results of Chapter 8.

3.1.1 The Wiener-Hopf factorization

Let us define the key ingredients necessary to state the Wiener-Hopf Factorization. We assume that X is a Lévy process with characteristic exponent $\Psi(z)$, and we define the running supremum S and infimum I processes by the equalities

$$S_t := \sup_{0\leq s\leq t} X_t, \quad \text{and} \quad I_t := \inf_{0\leq s\leq t} X_t$$

respectively. Now we define

$$\overline{G}_t := \sup\{s < t : X_s = S_t\}, \quad \text{and} \quad \underline{G}_t := \inf\{s < t : X_s = I_t\}.$$

As usual, we denote by $\mathbf{e}(q)$ an exponential random variable independent of the process X with parameter $q > 0$. Finally, define for $\theta,\ z \in \bar{\mathbb{C}}^+$ and $q > 0$ the functions

$$\Psi_q^+(\theta, z) := \exp\left(\int_{\mathbb{R}^+}\int_{\bar{\mathbb{R}}^+} (e^{i\theta t+izx} - 1)t^{-1}e^{-qt}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t\right), \text{ and} \tag{3.1}$$

$$\Psi_q^-(\theta, z) := \exp\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}^-} (e^{i\theta t-izx} - 1)t^{-1}e^{-qt}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t\right). \tag{3.2}$$

Theorem 3 (The Wiener-Hopf Factorization, Part 1)**.**

(i) The pairs $(\overline{G}_{\boldsymbol{e}(q)}, S_{\boldsymbol{e}(q)})$ *and* $(\boldsymbol{e}(q) - \overline{G}_{\boldsymbol{e}(q)}, S_{\boldsymbol{e}(q)} - X_{\boldsymbol{e}(q)})$ *are independent, infinitely divisible random variables.*[1]*.*

(ii) Further, $(\boldsymbol{e}(q), X_{\boldsymbol{e}(q)})$ *is infinitely divisible, and its characteristic exponent admits the factorization*

$$\mathbb{E}[e^{i\theta \boldsymbol{e}(q) + izX_{\boldsymbol{e}(q)}}] = \frac{q}{q - i\theta + \Psi(z)} = \Psi_q^+(\theta, z)\Psi_q^-(\theta, z) \tag{3.3}$$

where $\theta, z \in \mathbb{R}$ *and,*

$$\Psi_q^+(\theta, z) = \mathbb{E}[e^{i\theta \overline{G}_{\boldsymbol{e}(q)} + izS_{\boldsymbol{e}(q)}}], \qquad \text{and} \qquad \Psi_q^-(\theta, z) = \mathbb{E}[e^{i\theta \underline{G}_{\boldsymbol{e}(q)} + izI_{\boldsymbol{e}(q)}}].$$

We call $\Psi_q^+(\theta, z)$ *and* $\Psi_q^-(\theta, z)$ *the positive and negative Wiener-Hopf factors respectively.*

(iii) The factorization in (ii) is unique. That is, suppose we can find functions $f^+(\theta, z)$ *and* $f^-(\theta, z)$ *such that* $\mathbb{E}[e^{i\theta \boldsymbol{e}(q) + izX\boldsymbol{e}(q)}] = f^+(\theta, z)f^-(\theta, z)$ *for* $\theta, z \in \mathbb{R}$*. Further, suppose that* $f^+(\theta, z)$ *and* $f^-(\theta, z)$ *are the characteristic functions of infinitely divisible random variables, whose distributions are supported on* $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+$ *and* $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^-$ [2]*, and whose characteristic exponents have no drift. Then, these random variables are equal in distribution to* $(\overline{G}_{\boldsymbol{e}(q)}, S_{\boldsymbol{e}(q)})$ *and* $(\underline{G}_{\boldsymbol{e}(q)}, I_{\boldsymbol{e}(q)})$ *respectively and* $f^+(\theta, z) = \Psi_q^+(\theta, z)$ *while* $f^-(\theta, z) = \Psi_q^-(\theta, z)$ *for* θ, $z \in \bar{\mathbb{C}}^+$*.*

Proofs of the Wiener-Hopf factorization are available in [103, Chapter 9], [14, Chapter 6], [76, Chapter 6], and [50]. The history of this theorem dates back to so-called fluctuation theory for random walks. A collection of identities, similar to those in Theorem 3, relating the behaviour of a random walk and its extrema is proved in works by Spitzer [106, 107], Feller [43], and Borovkov [17, 18] to name a few. The name "Wiener-Hopf" derives from the 1957 paper by

[1] We did not define two, or d-dimensional infinitely divisible random variables in the Introduction. However, the same definition applies, as does the Lévy-Khintchine Formula. The characteristic exponent for a d-dimensional process takes the form: $\Psi(z) = i\langle a, z\rangle + \frac{1}{2}\langle z, Qz\rangle - \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{I}(|x| < 1)\Pi(\mathrm{d}x)$ where $\langle \cdot, \cdot \rangle$ is the dot-product, and Q is a symmetric nonnegative-definite $d \times d$ matrix. See [103] for the multi-dimensional case.

[2] Note this automatically implies that the infinitely divisible random variables have generating triples of the form $(a, 0, \Pi)_{h\equiv 0}$ where a is the drift.

Spitzer [106] in which the author shows that the limiting distribution function $F(x)$ of the maximum of a random walk satisfies

$$F(x) = \int_{\mathbb{R}^+} k(x-y)F(y)\mathrm{d}y$$

where $k(x)$ is a probability density function. This equation is a specific case of the Wiener-Hopf integral equation which can be solved by an analytic factoring method known as the Wiener-Hopf method.

Like the approach taken by Spitzer, most early work on the Wiener-Hopf factorization had an analytical flavour. However, the probabilistic approach taken by Pitman and Greenwood is responsible for the modern version of the Wiener-Hopf factorization for random walks [51] and for Lévy processes [50]. We give here a quick and informal overview of their approach for random walks, and then describe the analogous components for Lévy processes. We encourage the reader to consult Figure 3.1 while reading as we introduce a fair amount of notation which is best understood graphically. All quantities in this diagram represent specific realizations of random processes or variables; below, we will refer to the realizations and the random variables/processes by the same symbols.

We recall that a random walk, R, is a discrete time stochastic process of the form

$$R_n = \sum_{i=0}^{n} \xi_i,$$

where the ξ_i are i.i.d. random variables, which we assume for simplicity are continuous. We further make the simplifying assumption that the process oscillates, i.e. that $\limsup_{n\to\infty} R_n = -\liminf_{n\to\infty} R_n = \infty$ almost surely. These two conditions ensure that following each new maximum/minimum of R there must exist a next maximum/minimum strictly greater/smaller than the one before.

The central idea in [51] is to break the path of the random walk into excursions from the maximum. That is, suppose the process has reached a new maximum, i.e. a point R_n such that $R_n > R_i$, $0 \le i < n$, then the path of the process from time $n+1$ until a new maximum is reached is called an excursion from the maximum. The ith excursion lasts for $\mathcal{T}_i$ units of time; we will call this random variable the ith ***ascending ladder time***. The distance between the $(i-1)$th and ith maxima, the so-called ith ***ascending ladder height***, will be labeled

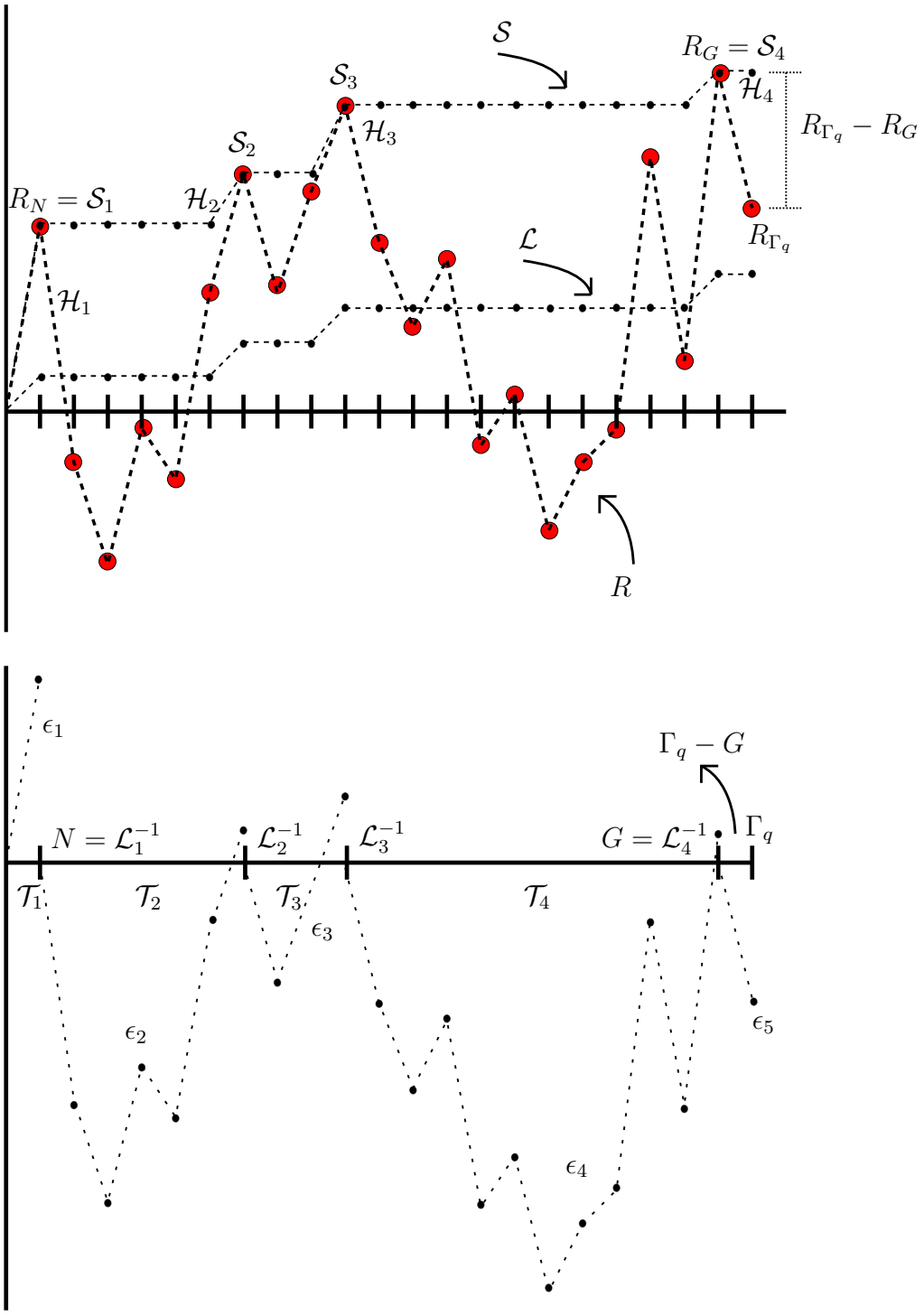


Figure 3.1: Path decomposition of a random walk up to a geometric time Γ_q. The upper figure shows realizations of R, $\mathcal{L}$, and the running supremum process $\mathcal{S}$. Below, we see the realization of the ladder time process $\mathcal{L}^{-1}$ and the excursion paths. Although we have drawn continuous paths to aid in visualization the reader should understand that all processes are in discrete time.

$\mathcal{H}_i$. Let $\mathcal{L}$ denote the process that counts the times at which $\mathcal{S}$, the running supremum process, is equal to R. Then, the inverse process, $\mathcal{L}^{-1}$, defined by $\mathcal{L}_n^{-1} := \min\{m \in \mathbb{N} : \mathcal{L}_m = n\}$ gives us the sum of the $\mathcal{T}_i$ and is labeled $\mathcal{L}^{-1}$. We will say the $(i+1)$th ***excursion***, which we now define formally to be the process $\epsilon_{i+1} := \left\{R_{\mathcal{L}_i^{-1}+k} - R_{\mathcal{L}_i^{-1}} : 0 < k \leq \mathcal{L}_{i+1}^{-1} - \mathcal{L}_i^{-1}\right\}$, is ***complete*** before some time n if $\mathcal{L}_{i+1}^{-1} \leq n$. Finally, let Γ_q be a geometric random variable with parameter $q > 0$ which is independent of R, and let G be the unique time at which R obtains its maximum on the set $\{0, \ldots, \Gamma_q\}$. We notice that since the time of the last completed excursion prior to Γ_q must coincide with G we have $G = \mathcal{L}_{\mathcal{L}_{\Gamma_q}}^{-1}$.

A critical conclusion of the analysis is that $\{\mathcal{L}_i^{-1}\}_{i\geq 1}$ is a collection of stopping times for the natural filtration of R so that by the strong Markov property (see Section 4.3.1), and the fact that random walks have independent and stationary increments, the excursion, ϵ_{i+1}, is independent of $\mathcal{F}_{\mathcal{L}_i^{-1}}$. In other words, the path of R is just the sum of the paths of independent excursions from the maximum. This has two important implications: the first is that (G, R_G) and $(\Gamma_q - G, R_{\Gamma_q} - R_G)$ are independent; the second is that the so-called ***ascending ladder process***, $(\mathcal{L}^{-1}, \mathcal{S})$, which is defined by $(\mathcal{L}_n^{-1}, \mathcal{S}_n) := \sum_{1\leq i\leq n}(\mathcal{T}_i, \mathcal{H}_i)$ is a strictly increasing random walk with $(\mathcal{T}_i, \mathcal{H}_i) \stackrel{d}{=} (N, R_N)$ where N represents the first time that R is strictly greater than 0.

Now we reason as follows: G must be equal to the sum of the $\mathcal{T}_i$ up to the last completed excursion prior to the time Γ_q. Likewise, R_G must be equal to the sum of the $\mathcal{H}_i$. Our knowledge of the ladder height process and the memoryless property of Γ_q lead us to conclude that $\mathbb{P}(\mathcal{T}_i \leq t, \mathcal{H}_i \leq h | \mathcal{L}_i^{-1} \leq \Gamma_q) = \mathbb{P}(N \leq t, R_N \leq h | N \leq \Gamma_q)$. That is, the joint distribution of the ith ladder time/height, given that the ith excursion is completed prior to Γ_q is just that of (N, R_N) conditioned on the event $\{N \leq \Gamma_q\}$. Further, we can show that these conditional ladder heights/times are still independent. Finally, the memoryless property dictates that $\mathbb{P}(\mathcal{L}_{\Gamma_q} > k) = \mathbb{P}(\sum_{0\leq i\leq k} \mathcal{T}_i < \Gamma_q) = \mathbb{P}(N < \Gamma_q)^k$, i.e. that the number of completed excursions before time Γ_q, namely $\mathcal{L}_{\Gamma_q}$, follows a geometric distribution with parameter $\mathbb{P}(N < \Gamma_q)$. We summarize our findings: (G, R_G) is equal in distribution to the sum of $\mathcal{L}_{\Gamma_q}$ i.i.d. random variables, in other words, to a random walk evaluated at geometric time $\mathcal{L}_{\Gamma_q}$. Since such a sum has an infinitely divisible distribution supported on $\mathbb{N}\cup\{0\}\times\bar{\mathbb{R}}^+$ without drift, it follows that (G, R_G) is infinitely divisible (with these properties), and by the same reasoning, so is (Γ_q, R_{Γ_q}).

A classic result from the theory of random walks known as the Duality Lemma (see pg. 394 in [43]) says that for any fixed n the ***reversed process***, that is the process R^* which

we define by

$$R_k^* = R_n - R_{n-k} \text{ where } k = 0, \ldots, n$$

has the same law as the random walk R stopped at time n. A consequence of the Duality Lemma is that $(\Gamma_q - G, R_{\Gamma_q} - R_G) \stackrel{d}{=} (D, R_D)$ where D is a random variable which represents the unique time at which R visits its minimum on the set $\{0, \ldots, \Gamma_q\}$. Carrying out the same analysis as above, except now with the *descending* ladder process, we arrive at the conclusion,

$$(\Gamma_q, R_{\Gamma_q}) \stackrel{d}{=} (G, R_G) + (D, R_D) \tag{3.4}$$

where each of the variables in (3.4) has an infinitely divisible distribution, and the variables on the right hand side of the equality are independent. This is essentially the discrete time version of Theorem 3 part (i) and the majority of part (ii). The statement on uniqueness follows from the fact that an infinitely divisible random variable whose characteristic exponent has neither drift nor Gaussian component is uniquely determined by its Lévy measure. For a more detailed description of the Wiener-Hopf factorization for random walks see [75].

The proof of the Wiener-Hopf factorization for Lévy processes follows the same strategy as in the random walk case. The major difference is that it is no longer apparent how to describe a process $\mathcal{L}$ that counts the times at which our Lévy process X reaches a new maximum. This is because the set of points at which this occurs is (in general) no longer discrete so the notion of "counting" needs to be redefined. However, it is possible to demonstrate the existence of a process L, called the ***local time at the maximum*** or ***local time*** for short, that is strictly increasing and changes only on the closure of the set $\{t \geq 0 : X_t = S_t\}$ (see [76, pg. 140] for a precise definition). Unsurprisingly, the form of L depends on the first time $t > 0$ in which X enters the half plane $[0, \infty)$. Processes, for which the entry is almost surely immediate – for which we say the point 0 is ***regular*** for $[0, \infty)$ – have continuous versions of local time, while those for which it is not – i.e. 0 is ***irregular*** for $[0, \infty)$ – have right-continuous versions; in both cases versions of the local time are unique up to a multiplicative constant. For finite variation processes, for which 0 is regular for $[0, \infty)$ but irregular for $(-\infty, 0)$, we have an obvious continuous time analogue of our counting process $\mathcal{L}$. Namely, if L is a local time of such a process, then it has the form $L_t = a \int_0^t \mathbb{I}(X_u = S_u) \mathrm{d}u$ where $a > 0$ is a constant (see Theorem 6.8 in [76]).

Despite some differences, the important pieces of the proof for Lévy processes rely on the

same key ideas demonstrated in our discussion of random walks. Our process still obeys the strong Markov property, and has stationary and independent increments. Further, the random variable $\mathbf{e}(q)$ is memoryless, a Lévy process evaluated at an independent exponential time has an infinitely divisible distribution, and we can establish a Duality Lemma for Lévy processes (see [76, pg. 73-75]). Critically, we can also break paths of X into independent excursions from the maximum using the continuous time analog of the ascending ladder process, which we will call (L^{-1}, H). For a version of local time this is the bivariate process defined by

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty, \\ \infty & \text{otherwise.} \end{cases}$$

$$H_t := \begin{cases} X_{L_t^{-1}} & \text{if } t < L_\infty, \\ \infty & \text{otherwise.} \end{cases}$$

Instead of an increasing random walk, as was the case for $(\mathcal{L}^{-1}, \mathcal{S})$ in the discrete time scenario, (L^{-1}, H) is a (possibly killed) bivariate subordinator [3]. A similar result holds for the descending ladder process, $(\hat{L}^{-1}, \hat{H})$, the process based on the local time at the minimum, which we can construct from a version of local time for $-X$. The ladder processes have a number of useful applications. In fact, we may even express the Wiener-Hopf factors of X in terms of the joint Laplace transforms of (L^{-1}, H), and $(\hat{L}^{-1}, \hat{H})$, which we denote by $\kappa(\alpha, \beta)$ and $\hat{\kappa}(\alpha, \beta)$ respectively. For α, $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta) \geq 0$ we define $\kappa(\alpha, \beta)$ by

$$e^{-\kappa(\alpha,\beta)} := \mathbb{E}[e^{-(\alpha L_1^{-1} + \beta H_1)} \mathbb{I}(1 < L_\infty)],$$

and use an analogous definition for $\hat{\kappa}(\alpha, \beta)$ in terms of $(\hat{L}^{-1}, \hat{H})$. This leads us to provide the following addendum to Theorem 3.

Theorem 3 (The Wiener-Hopf Factorization, Part 2)**.**

(iv) For α, $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta) \geq 0$ we have the following identity for the Wiener-Hopf factors

$$\Psi_q^+(i\alpha, i\beta) = \frac{\kappa(q, 0)}{\kappa(q + \alpha, \beta)},$$
$$\Psi_q^-(i\alpha, -i\beta) = \frac{\hat{\kappa}(q, 0)}{\hat{\kappa}(q + \alpha, \beta)}.$$

[3] In the two dimensional case, this means a process whose paths increase in each coordinate almost surely.

(v) For some constant $k' > 0$, which depends on the version of local time, we have

$$k'\Psi(z) = \kappa(0, -iz)\hat{\kappa}(0, iz), \quad z \in \mathbb{R}.$$

Although we have presented the ideas behind the excursion theoretic proof of the Wiener-Hopf factorization, it is important to note that this is not the only method of proving this theorem. While excursion theory provides an intuitively satisfying proof based on probabilistic arguments, the rigorous proof of Theorem (3) by these techniques is quite involved and requires knowledge of some fairly sophisticated mathematics, for example the theory of local times for Markov processes. Another possibility is to derive the Wiener-Hopf factorization for Lévy processes by extending it first from random walks to compound Poisson processes and then to general Lévy processes. This is the approach taken in [103], but it is also laborious. A recent paper [67] proves the factorization (in almost complete generality) in a fairly direct manner using primarily analytic techniques. That is, the author largely forgoes the probabilistic approach and solves an integral equation using Wiener-Hopf techniques and other results from complex analysis. We conclude this section by adding to Theorem 3 once more with a further uniqueness result from this paper. First, we introduce the so-called spatial Wiener-Hopf factorization which we obtain by taking the limit in (3.3) as $\theta \to 0$. For $z \in \mathbb{R}$ we get

$$\frac{q}{q + \Psi(z)} = \mathbb{E}[e^{izS_{\mathbf{e}(q)}}]\mathbb{E}[e^{izI_{\mathbf{e}(q)}}] = \phi_q^+(z)\phi_q^-(z),$$

where, $\phi_q^+(z) := \Psi_q^+(0, z)$ for $z \in \bar{\mathbb{C}}^+$ and $\phi_q^-(z) := \Psi_q^-(0, -z)$ for $z \in \bar{\mathbb{C}}^-$. Substituting in $-iz$ for z in this equation allows us to express the Wiener-Hopf factorization in terms of the Laplace exponent, that is,

$$\frac{q}{q - \psi(z)} = \mathbb{E}[e^{zS_{\mathbf{e}(q)}}]\mathbb{E}[e^{zI_{\mathbf{e}(q)}}] = \varphi_q^+(z)\varphi_q^-(z),$$

where $\varphi_q^+(z) := \phi_q^+(-iz)$ for all $\mathrm{Re}(z) \leq 0$, and $\varphi_q^-(z) := \phi_q^-(-iz)$ for all $\mathrm{Re}(z) \geq 0$. The remaining results and theorems presented in this work concern only the spatial Wiener-Hopf factorization. Therefore, from now on, any reference to the Wiener-Hopf factors, is in fact a reference to $\phi_q^+(z)$ and $\phi_q^-(z)$ or equivalently to $\varphi_q^+(z)$ and $\varphi_q^-(z)$. Now, we can provide a final addendum to Theorem 3 (see Theorem 1, (f) in [67])

Theorem 3 (The Wiener-Hopf Factorization, Part 3)**.**

(vi) Assume there exist two functions $f^+(z)$ and $f^-(z)$ such that $f^\pm(0) = 1$, $f^\pm(z)$ is analytic in $\mathbb{C}^\pm$, $f^\pm(z)$ is continuous without roots in $\bar{\mathbb{C}}^\pm$, and $z^{-1}\log(f^\pm(z)) \to 0$ as $z \to \infty$, $z \in \bar{\mathbb{C}}^\pm$. If

$$\frac{q}{q+\Psi(z)} = f^+(z)f^-(z), \ z \in \mathbb{R}$$

then $f^\pm(z) = \phi_q^\pm(z)$ for all $z \in \bar{\mathbb{C}}^\pm$.

3.2 Determining the Wiener-Hopf Factors

3.2.1 Three techniques demonstrated via an example

We outline here the most recent progress in determining the Wiener-Hopf factors for various families of Lévy processes. Deriving explicit or even semi-explicit representations is a difficult task, and poses an open problem for many processes. However, the last ten years have seen a series of new results which we summarize here. It is difficult to give proofs, because these are usually technical, long, and specific to the process. However, from a macroscopic point of view, there are really only three different methods employed in proving all of the theorems below. We demonstrate these approaches using the hyper-exponential process as an example. Recall that such a process has a Laplace exponent which is a rational function on $\mathbb{C}$ of the form

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z\sum_{n=1}^{N}\frac{a_n}{\rho_n - z} - z\sum_{n=1}^{\hat{N}}\frac{\hat{a}_n}{\hat{\rho}_n + z}.$$

We assume, for simplicity, that $\sigma > 0$ and also recall the interlacing condition (2.13) of the poles, $\{-\hat{\rho}_n\}_{1\le n\le \hat{N}}, \{\rho_n\}_{1\le n\le N}$, and the solutions, $\{-\hat{\zeta}_n\}_{1\le n\le \hat{N}}, \{\zeta_n\}_{1\le n\le N}$, of the equation $\psi(z) = q$.

Approach 1: Identifying infinitely divisible distributions

This is probably the most naïve approach, and without a great deal of luck or intuition it is difficult to apply in many scenarios. The idea is to simply factor the function $q/(q-\psi(z))$ in such a way that we can identify the Laplace transforms of two infinitely divisible distributions

with support on $\bar{\mathbb{R}}^+$ and $\bar{\mathbb{R}}^-$ respectively. With this in mind, let us consider the hyper-exponential process and define the functions:

$$f^+(z) := \frac{1}{1-\frac{z}{\zeta_1}} \prod_{n=1}^{N} \frac{1-\frac{z}{\rho_n}}{1-\frac{z}{\zeta_{n+1}}}, \quad \text{and} \quad f^-(z) := \frac{1}{1+\frac{z}{\hat{\zeta}_1}} \prod_{n=1}^{\hat{N}} \frac{1+\frac{z}{\hat{\rho}_n}}{1+\frac{z}{\hat{\zeta}_{n+1}}}. \tag{3.5}$$

It is easy to verify that the rational function $q/(q-\psi(z))$ can be factored in the form $f^+(z)f^-(z)$. Now, employing a partial fraction decomposition, we may write $f^+(z)$ and $f^-(z)$ as,

$$f^+(z) = \sum_{n=1}^{N+1} \frac{\zeta_n \beta_n}{(\zeta_n - z)}, \quad \text{and} \quad f^-(z) = \sum_{n=1}^{\hat{N}+1} \frac{\hat{\zeta}_n \hat{\beta}_n}{(\hat{\zeta}_n + z)},$$

where,

$$\beta_n := \prod_{k \neq n} \frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\zeta_k}}, \quad \text{and} \quad \hat{\beta}_n := \prod_{k \neq n} \frac{1-\frac{\hat{\zeta}_n}{\hat{\rho}_k}}{1-\frac{\hat{\zeta}_n}{\hat{\zeta}_k}}.$$

From the interlacing property, it is apparent that the $\{\beta_n\}_{1\leq n\leq N+1}$ and $\{\hat{\beta}_n\}_{1\leq n\leq \hat{N}+1}$ are all positive, and setting $z=0$ we notice that $\sum_{1\leq n\leq N+1}\beta_n = \sum_{1\leq n\leq \hat{N}+1}\hat{\beta}_n = 1$. Now it is clear that we have chosen the correct method of factoring $q/(q-\psi(z))$. That is, we can identify $f^+(z)$ as the Laplace transform of a random variable ξ^+ whose distribution is equal to a finite mixture of exponential distributions with parameters $\{\zeta_n\}_{1\leq n\leq N+1}$. And, likewise, $f^-(z)$ is the Laplace transform of a random variable ξ^-, where $-\xi^-$ is a finite mixture of exponential distributions with parameters $\{\hat{\zeta}_n\}_{1\leq n\leq \hat{N}+1}$. Since these are both infinitely divisible random variables without drift, whose distributions are supported on $\bar{\mathbb{R}}^+$ and $\bar{\mathbb{R}}^-$ respectively, we may, by Theorem 3 (iii) conclude that we have established the Wiener-Hopf factorization and that $\xi^+ \stackrel{d}{=} S_{\mathbf{e}(q)}$ and $-\xi^- \stackrel{d}{=} I_{\mathbf{e}(q)}$.

Approach 2: Analytic factoring

Before considering this approach in the context of the hyper-exponential process, we need to describe it properly. This is not so easy, since the technique varies somewhat for different processes. There is, however, a common first step which identifies the method.

Let us assume that X is a Lévy process whose Laplace exponent $\psi(z)$ is meromorphic for $\mathrm{Re}(z) \geq 0$ and that all solutions of $q-\psi(z)$ with positive real part are bounded away from the line $i\mathbb{R}$. As the Laplace transform of a positive random variable $\varphi_q^+(z)$ is continuous for

$\mathrm{Re}(z) \leq 0$ and analytic for $\mathrm{Re}(z) < 0$. Further, the identity (Theorem 3 Part 2 (iv))

$$\varphi_q^+(z) = \frac{\kappa(q,0)}{\kappa(q,-z)} \tag{3.6}$$

and the fact that $\kappa(q,-z)$ is analytic for $\mathrm{Re}(z) < 0$ show that $\varphi_q^+(z)$ has no zeros in the left half-plane. By the same reasoning $\varphi_q^-(z)$ has no zeros in the right half-plane, and it is clearly analytic there. Combining these facts about $\varphi_q^+(z)$ and $\varphi_q^-(z)$, our assumption about $\psi(z)$, and the Wiener-Hopf identity

$$\varphi_q^+(z) = \frac{q}{(q-\psi(z))\varphi_q^-(z)}, \quad z \in i\mathbb{R},$$

we see that we may extend $\varphi_q^+(z)$ to a meromorphic function on $\mathbb{C}$ which is equal to $q/((q-\psi(z))\varphi_q^-(z))$ for $\mathrm{Re}(z) \geq 0$ (this is essentially due to Morera's theorem). Therefore, the zeros and singularities of $\varphi_q^+(z)$ lie in the right half-plane and are equal to the zeros and singularities of the the function $q/(q-\psi(z))$ there. If we had initially assumed that $\psi(z)$ was meromorphic for $\mathrm{Re}(z) \leq 0$, and that the solutions of $q-\psi(z)$ with negative real part were bounded away from $i\mathbb{R}$ we would have obtained an analogous result for $\varphi_q^-(z)$.

The extension of $\varphi_q^+(z)$ (resp. $\varphi_q^-(z)$) to $\mathbb{C}$ and the identification of the zeros and singularities as those of $q/(q-\psi(z))$ are the defining elements of the analytic factoring method. The necessary assumption is that $\psi(z)$ is meromorphic on one, or both half-planes and that the solutions of $q-\psi(z)$ are bounded away from the line $i\mathbb{R}$.

From here, the approaches for determining $\varphi_q^+(z)$ vary somewhat. However, the common idea in all cases is to establish a representation of the function $\varphi_q^+(z)$ as a product (with potentially infinitely many terms) of its zeros and singularities. Since these must coincide with the zeros and singularities of $q/(q-\psi(z))$, our problem will be reduced to finding zeros and singularities of this latter, known function.

The specific approach for the hyper-exponential case relies on the fact that $\psi(z)$ is a rational function, so the $q/(q-\psi(z))$ is easily factored into the form $f^+(z) \times f^-(z)$. Before using this fact, we remark that due to (3.1) and (3.3) we can identify $\varphi_q^+(z)$ as the Laplace transform of a subordinator without drift with Lévy measure $\Pi(\mathrm{d}x) = \int_{\mathbb{R}^+} t^{-1}e^{-qt}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t$. Similarly, we can identify $\varphi_q^-(z)$ as the Laplace transform of the dual of a subordinator. A slight modification of Proposition 2 in [14] then shows that $z^{-1}\log(\varphi_q^+(z)) \to 0$ as $z \to \infty$ for

$\text{Re}(z) \leq 0$, with the analogous result for $\varphi_q^-(z)$ for $\text{Re}(z) \geq 0$. Now we define the function

$$F(z) = \begin{cases} \frac{f^+(z)}{\varphi_q^+(z)} & \text{Re}(z) \leq 0 \\ \frac{\varphi_q^-(z)}{f^-(z)} & \text{Re}(z) \geq 0 \end{cases},$$

which is easily verified to be an entire function, and therefore $\log(F(z))$ is also an entire function [4]. Since $z^{-1}\log(f^{\pm}(z)) \to 0$, as $z \to \infty$ it is clear that $F(z)$ satisfies the same asymptotic condition. Applying Cauchy's estimates (pg. 73 in [34]) we see that $\log(F(z))$ must be a constant function on $\mathbb{C}$, and evaluating at 1, we see this constant must be 0. Therefore, $\varphi_q^+(z) \equiv f^+(z)$ on $\text{Re}(z) \leq 0$ and $\varphi_q^-(z) \equiv f^-(z)$ on $\text{Re}(z) \geq 0$.

Remark

The reader may recognize that we have used essentially the criteria of Theorem 3, Part 3 to determine the Wiener-Hopf factors. This is correct: In the above discussion we use the same techniques as those used to prove Theorem 3, Part 3 in [67]. It is clear that this approach needlessly is complicated for the hyper-exponential process. The true power of Approach 2 is that we can use it even when $q/(q-\psi(z))$ cannot be directly factored into the product of two functions $f^+(z)$ and $f^-(z)$ which clearly satisfy the criteria of Theorem 3, Part 3. See for example the results and discussion in Section 3.2.3.

Approach 3: Integral representation

The final approach involves writing the Wiener-Hopf factors using an alternate (to 3.1) integral representation and then solving the integral. This representation is given in the following theorem.

Theorem 4 (Theorem 1(b) in [67]). *For any Lévy process whose Lévy measure satisfies*

$$\int_{\mathbb{R}\setminus[-1,1]} |x|^{\epsilon}\Pi(\mathrm{d}x) < \infty \tag{3.7}$$

for at least one $\epsilon \in (0,1)$ *we have the following identity:*

$$\phi_q^{\pm}(z) = \exp\left(\pm\frac{z}{2\pi i}\int_{\mathbb{R}} \log\left(\frac{q}{q+\Psi(u)}\right)\frac{\mathrm{d}u}{u(u-z)}\right), \quad z \in \mathbb{C}^{\pm}. \tag{3.8}$$

[4] Here we no longer necessarily intend $\log(f(z))$ to represent the composition of the principal branch of the logarithm with $f(z)$, but rather an analytic function $g(z)$ such that $\exp(g(z)) = f(z)$. See X.5 in [102] for a discussion and proof of existence.

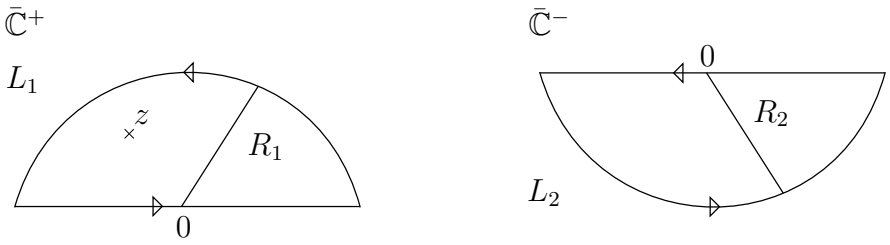


Figure 3.2: Contours of integration

Condition (3.7) is a mild one, and is satisfied by all Lévy processes in this work. It fails only in cases where the Lévy measure has very heavy tails. One may prove Theorem 4 by showing that the right-hand side of (3.8) satisfies the conditions of Theorem 3 (vi).

To evaluate the integral in (3.8) for a hyper-exponential process, we may again let the zeros and singularities of the function $q/(q+\Psi(z))$ be our guide. Specifically, let us factor the function $q/(q+\Psi(z))$ according to (3.5) and re-write (3.8) as

$$\phi_q^+(z) = \exp\left(\frac{z}{2\pi i}\int_{\mathbb{R}} g^+(u) + g^-(u)\mathrm{d}u\right),$$

where,

$$g^{\pm}(u) = \frac{\log(f^{\pm}(iu))}{u(u-z)}.$$

Now we use the standard techniques of contour integration to complete our task. Consider the contours L_1 and L_2 in Figure 3.2. For a fixed $z \in \mathbb{C}^+$ and R_1 sufficiently large, the function $g^+(u)$ is analytic [5] inside and on the contour L_1 except for a simple pole at z. On the other hand, the function $g^-(u)$ is analytic on $\bar{\mathbb{C}}^-$; in particular, $g^-(u)$ is analytic on and inside L_2. Therefore, according to the residue theorem,

$$\int_{L_1} g^+(u)\mathrm{d}u = 2\pi i \log(f^+(iz))/z,$$

while Cauchy's integral theorem ensures that the integral of $g^-(u)$ over the contour L_2 vanishes. The reader may verify that as $R_1, R_2 \to \infty$ the integrals of $g^+(u)$ and $g^-(u)$ on the semi-circular contours also vanish. This shows that $\phi_q^+(z) = f^+(iz)$ as was expected.

[5] Technically speaking, we have a removable singularity at the origin. Rather than introduce new notation we write $g^{\pm}(u)$ for the analytic continuation of our original functions.

3.2.2 Stable processes

Recall from Chapter 2 that we are interested in stable processes with parameters $(\alpha, \rho) \in \mathcal{A}$ where

$$\mathcal{A} = \{\alpha \in (0,1), \rho \in (0,1)\} \cup \{\alpha = 1, \rho = \tfrac{1}{2}\} \cup \{\alpha \in (1,2), \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}. \tag{3.9}$$

Since we make no restrictions on the behaviour of the positive or negative jumps of the processes in this section, we may concentrate only on the positive Wiener-Hopf factor $\varphi_q^+(z)$, as we may obtain the corresponding information on $\varphi_q^-(z)$ from the dual processes. Also, by the self-similarity property of stable processes we have

$$\varphi_q^+(z) = \varphi_1^+(zq^{-1/\alpha}),$$

which shows that we may limit ourselves to the case $q = 1$ without loss of generality. For Section 3.2.2, we will therefore define $\varphi(z) := \varphi_1^+(-z)$ and give all results in terms of $\varphi(z)$. This simplifies notation, and adheres to the conventional notation for stable processes.

We will make a few more useful definitions to aid us in this section. First, we note that a spectrally positive stable process must satisfy the property $\rho - 1 = -\alpha^{-1}$. Motivated by this, we'll define the class of processes $\mathcal{C}_{k,l}$, first introduced by Doney in [38], as the collection of stable processes whose parameters satisfy the equation

$$\rho + k = l/\alpha, \quad k, l \in \mathbb{Z}. \tag{3.10}$$

The classes $\{\mathcal{C}_{k,l}\}$ are often called the Doney classes. Next, for $n \in \mathbb{N}$ we define the ***q-Pochhammer symbol*** as

$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k),$$

and $(a;q)_0 = 1$. For $|q| < 1$ we may also define $(a;q)_\infty := \prod_{k\geq 0}(1-aq^k)$. Finally, for $z \geq 0$ we will refer to the following expression

$$-\frac{\sin(\pi\rho)}{\pi}\int_{\mathbb{R}^+}\frac{\log(1+(zu)^\alpha)}{u^2 + 2u\cos(\pi\rho) + 1}\mathrm{d}u,$$

as ***Darling's integral*** .

In the literature, Wiener-Hopf factorizations for stable processes are obtained exclusively via Approach 3. For symmetric processes ($\rho = 1/2$) Darling [36] showed that Darling's integral is equal to $\log(\varphi(z))$ for positive real z, and the same result was established in the general setting by Heyde [57]. It is not too difficult to show, essentially by substituting the expression for $\Psi(z)$ for stable processes (see (2.7)) into the integral in (3.8) and invoking an argument of "changing the contour of integration" (see Chapter 5 for details of this procedure), that evaluating Darling's integral is equivalent to using Approach 3. The first successful evaluation of Darling's integral as a means of obtaining the Wiener-Hopf factorization is due to Doney [38], who was able to do so precisely for processes in the Doney classes. The main result from this work is given in Theorem 5.

Theorem 5 (Theorem 2 in [38]). *If $X \in \mathcal{C}_{k,l}$, then for $|\arg(z)| < \pi$*

$$\varphi(z) = \begin{cases} \dfrac{(z^\alpha(-1)^{1-l}q^{(1-k)/2};q)_k}{(z(-1)^{(1-k)}\tilde{q}^{(1-l)/2};\tilde{q})_l}, & \text{if } l > 0, \\ \dfrac{(z(-1)^{1+k}\tilde{q}^{(1+l)/2};\tilde{q})_{|l|}}{(z^\alpha(-1)^{1+l}q^{(1+k)/2};q)_{|k|}}, & \text{if } l < 0, \end{cases}$$

$$\text{where,}$$

$$q = e^{2\pi i\alpha} \quad \text{and} \quad \tilde{q} = e^{-2\pi i\alpha}.$$

Theorem 5 was established in 1987, but it is only recently that we have seen further development in this area. In [68] Kuznetsov shows via Approach 3, that $\frac{\mathrm{d}}{\mathrm{d}z}\log(\varphi(z))$ can be expressed in terms of a certain elliptic-like function. By studying this function's properties the author is able to derive an explicit expression for $\varphi(z)$ for a general stable process. The expression simplifies considerably for processes corresponding to certain subsets of $\mathcal{A}$. Just as Theorem 5 applies only to the Doney classes, membership in these subsets depends crucially on the arithmetic properties of α. We will encounter this dependence again when we discuss the density of the supremum process later in this section and in Chapter 7. While this dependence is apparent, its probabilistic interpretation – for example the probabilistic properties of processes belonging to the Doney classes – is still an open question. To give the first result from [68] we need to define the Clausen function. For $\theta \in \mathbb{R}$ the ***Clausen function*** is given by

$$\mathrm{Cl}_2(\theta) := \sum_{n\geq 1} \frac{\sin(n\theta)}{n^2}.$$

We may alternatively define $\mathrm{Cl}_2(\theta)$ as the imaginary part of the dilogarithm function, $\mathrm{Li}_2(z)$, evaluated at $e^{i\theta}$. See Appendix A or [82] for more information on these functions. With this

we may state the first explicit expression for the Wiener-Hopf factors valid for rational α.

Theorem 6 (Theorem 2 in [68]). *Assume that $\alpha = \frac{m}{n}$ where m and n are coprime natural numbers. Define*

$$\theta := \begin{cases} \cot^{-1}(\cot(\pi m\rho) + (-1)^{mn} z^m \sin(\pi m\rho)^{-1}), & \text{if } m\rho \notin \mathbb{Z}, \\ 0, & \text{if } m\rho \in \mathbb{Z}. \end{cases}$$

Then, for $z > 0$

$$\begin{aligned} \varphi(z) = \exp\left(\frac{1}{2\pi mn}(\mathrm{Cl}_2(2\theta) - \mathrm{Cl}_2(2\pi m\rho) - \mathrm{Cl}_2(2\theta - 2\pi m\rho)\right) \\ \times (1 + (-1)^{mn} 2\cos(\pi m\rho) z^m + z^{2m})^{-\rho/(2n)} \\ \times \prod_{k=0}^{n-1} (1 + 2\cos(\pi\alpha(\rho + 2k + 1)) z^{\alpha} + z^{2\alpha})^{(n-2k-1)/(2n)} \\ \times \prod_{j=0}^{m-1} \left(1 + 2\cos\left(\frac{\pi}{\alpha}(\alpha\rho + 2j + 1)\right) z + z^2\right)^{(m-2j-1)/(2m)} \end{aligned}$$

Having given a result for all rational parameters, we consider now the irrational numbers. Here we encounter an instance in which the arithmetic properties of the parameters play an important role. It turns out that irrational numbers for which there are too many good rational approximations form an exception. To make this precise, we will define the set $\mathcal{L}$ to consist of irrational numbers x, where x satisfies

$$\left| x - \frac{p}{q} \right| < \frac{1}{b^q}$$

for some $b > 1$, and infinitely many coprime integers p and q. One may prove, (see for example [58]) that the set $\mathcal{L}$ is closed under multiplication and addition by the rational numbers and is thus dense in $\mathbb{R}$. It is a subset of the Liouville numbers, which are defined by the weaker condition: x is a Liouville number if for all $n \geq 1$ the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^n} \tag{3.11}$$

is satisfied by infinitely many integers p and q. Since the Liouville numbers are a set of Lebesgue measure 0 and Hausdorff dimension 0, $\mathcal{L}$ inherits these same properties. In this sense numbers in the set $\mathcal{L}$ are somewhat rare. Theorem 5 in [68] shows that for any $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ we may write $\log(\varphi(z))$ as an infinite sum involving the quotient of sine functions. The de-

nominator in this quotient is a term of the form $k\sin(\pi k\alpha)$ for $k \in \mathbb{N}$. We see that the series cannot be defined for rational α since in these cases we will eventually end up with division by 0. And, for $\alpha \in \mathcal{L}$ it turns out that we are not able to bound the terms away from zero in order to ensure the series' convergence.

The final result we give from [68] is a completely general result for the Wiener-Hopf factorization expressed in terms of the double-gamma function $G(z;\tau)$, where $|\arg(\tau)| < \pi$ and $z \in \mathbb{C}$. For more information on the double-gamma function see Appendix A, pg. 1040–1041 in [68], or [10] and [11].

Theorem 7 (Theorem 4 in [68]). *For* $(\alpha,\rho) \in \mathcal{A}$, *and* $|\arg(z)| < \pi$

$$\varphi(z) = (2\pi\sqrt{z})^{\alpha\rho}\frac{G(1/2+\alpha/2(1+\rho+\log(z)/(\pi i));\alpha)}{G(1/2+\alpha/2(1-\rho+\log(z)/(\pi i));\alpha)} \times \frac{G(1/2+\alpha/2(1+\rho-\log(z)/(\pi i));\alpha)}{G(1/2+\alpha/2(1-\rho-\log(z)/(\pi i));\alpha)}.$$

Since we may also express the double-gamma function as an infinite product of gamma functions, as a corollary to Theorem 7 we have an infinite product representation of $\varphi(z)$ for any stable process with the admissible parameters (see Corollary 3 in [68]).

The density of the supremum of a stable process

A subject closely related to the Wiener-Hopf factorization is that of the distribution of the supremum process S. It is easy to see that for any Lévy process

$$\varphi_q^+(z) = q\int_{\mathbb{R}^+} e^{-qt}\mathbb{E}[e^{zS_t}]\mathrm{d}t.$$

Therefore we should be able to recover information about the supremum process by inverting the Laplace transform. It turns out that for stable processes a more fruitful approach is to consider the Mellin transform of $\varphi(z)$. Via the analytic properties $\varphi(z)$ inherits from the double-gamma function it is possible to derive an explicit expression for the Mellin transform (see [68]), which leads to several different asymptotic and convergent series expressions for the density $p(x)$ of S (see [58, 68, 70]). We discuss this theory in more detail in Chapter 7, where we encounter the set $\mathcal{L}$ and the importance of the arithmetic properties of the parameters (α,ρ) again. We also note that for spectrally one-sided processes we have complete information about the distribution of the extrema due to the work of Bingham [16], Doney [39], Bernyk, Dalang and Peskir [13] and Patie [90]. We will not consider one-sided stable processes further in this work.

3.2.3 Processes with bounded positive jumps and processes with positive jumps of rational transform

In this section we consider two types of processes in which we constrain the behaviour of the positive jumps. In the first case, we focus on processes with arbitrarily defined negative jumps and positive jumps of rational transform, and in the second case we set an upper bound for the positive jumps while allowing the negative jumps to behave arbitrarily. Reducing the generality serves not only to develop a theoretical result, but also has practical value. The primary benefit is that both types of processes can be used to approximate more general processes quite easily. Additionally, there is a direct application for processes with bounded jumps in actuarial science. We may use the dual of such a process to model the wealth of an insurance company, with the negative jumps representing the claims. The wealth of an insurance company that is reinsured above a certain claim amount is well represented by such a process.

We briefly define both types of processes formally and then give the main results. From Chapter 2 the reader will be familiar with processes of rational transform. From this and the description just given we know that a ***process with positive jumps of rational transform*** is defined by its Lévy measure, which has the form $\Pi(dx) = \Pi^-(\mathrm{d}x) + \Pi^+(\mathrm{d}x)$ where Π^- may be any Lévy measure concentrated on $\mathbb{R}^-$ and Π^+ is an absolutely continuous Lévy measure on $\mathbb{R}^+$ with density

$$\pi^+(x) = \mathbb{I}(x > 0)\lambda \sum_{n=1}^{N} \sum_{j=1}^{M_n} c_{nj}(\rho_n)^j \frac{x^{j-1}}{(j-1)!} e^{-\rho_n x}. \tag{3.12}$$

We see that if X is such a process, then we may write $X = X^+ + X^-$ where X^+ is a process with only positive jumps defined by (3.12), and X^- has only negative jumps. To define a process with bounded positive jumps we first need to define the ***positive tail*** of a Lévy measure $\Pi(\mathrm{d}x)$ as: $\bar{\Pi}^+(x) := \Pi((x, \infty))$ for $x > 0$. We say a process with Lévy measure $\Pi(\mathrm{d}x)$ has ***bounded positive jumps*** if there is a $0 < k^+ < \infty$ such that

$$k^+ := \inf\{x > 0 : \bar{\Pi}^+(x) = 0\}.$$

Now we turn our attention to the Wiener-Hopf factorizations for these processes, both of which are obtained using a version of Approach 2 [6]. In the case of processes with positive

[6] The original proof of Theorem 8 is due to [83] and proceeds using Approach 3. More recently it was proved in [46] using Approach 2.

jumps of rational transform, the approach is similar to the approach for hyper-exponential processes.

Theorem 8 (Theorem 2.2 in [83]). *The positive Wiener-Hopf factor of a process with positive jumps of rational transform has the following form*

$$\varphi_q^+(z) = \prod_{n=1}^{N} \left(\frac{\rho_n - z}{\rho_n} \right)^{M_n} \prod_{j=1}^{J} \left(\frac{\zeta_j}{\zeta_j - z} \right)^{K_j}, \tag{3.13}$$

where $\{\zeta_j\}_{1 \le j \le J}$ *are the* J *distinct roots of the equation* $\psi(z) = q$ *in the plane* $\mathrm{Re}(z) > 0$ *and* $\{K_j\}_{1 \le j \le J}$ *their multiplicities.*

For practical application, we will need to have more information (e.g. location, multiplicity) about the roots $\{\zeta_j\}_{1 \le j \le J}$. For processes with both positive and negative jumps of rational transform, Theorem 2 gives us complete information about multiplicity. For processes with arbitrary negative jumps we have similar information. Namely, we know that ζ_1 is real and has multiplicity 1. If M_n denotes the multiplicity of n^{th} pole ρ_n, and $P = M_1 + \ldots + M_N$ denotes the total pole count, then the total root count, $K = 1 + K_2 + \ldots + K_J$ does not depend on q and is related to the pole count. Specifically, if X is a process with positive jumps of rational transform and if $-X^-$ is a subordinator then $P = K$, otherwise $K = P + 1$. Unfortunately, beyond the fact that the roots are located in the positive half-plane we do not have specific information about their location. We briefly discuss root finding algorithms in Chapter 5.

As a corollary to Theorem 8 we may also derive a generalized density (see Corollary 2.1 in [83]) for $S_{\mathbf{e}(q)}$, which is "generalized" in the sense that the distribution has an atom at 0 in the case where $-X^-$ is a subordinator. Additionally, Theorem 8 also holds for $q = 0$ provided that $\mathbb{E}[X_1] < 0$. Finally, the reader should note, that the results of [83] generalize earlier results on Lévy process with positive phase-type jumps; see, for example, [7, 88].

Now we shift our attention to processes with bounded positive jumps. Again, the approach to finding the Wiener-Hopf factors is via analytic factoring. The boundedness of the positive jumps implies that the function $f(z) = \kappa(q, iz)$ belongs to a special class of functions known as the Cartwright class. A property of such functions is that they have infinitely many zeros, and admit an infinite product representation in terms of linear factors of these zeros. Via the standard argument, we know that the zeros of $k(q, -z)$ on the right half-plane are just those of $q - \psi(z)$ there; these are the key ideas underlying the proof of the following theorem which can be found in [73].

Theorem 9 (Theorem 1 in [73]). *Assume that $q > 0$, and let $\psi(z)$ be the Laplace exponent of a process with bounded positive jumps. The equation $\psi(z) = q$ has a unique positive solution ζ_0 and infinitely many solutions in $\mathcal{Q}_1 := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$, denoted $\{\zeta_n\}_{n\geq 1}$. Assume that ζ_n are arranged in the order of increase of absolute value. The following statements are true:*

- *ζ_0 has multiplicity one and $\mathrm{Re}(\zeta_n) \geq \zeta_0$ for all $n \geq 1$.*
- *The Wiener-Hopf factors can be identified as follows: for $\mathrm{Re}(z) \geq 0$*

$$\varphi_q^+(-z) = e^{\frac{kz}{2}} \left(1 + \frac{z}{\zeta_0}\right)^{-1} \prod_{n\geq 1} \left(1 + \frac{z}{\zeta_n}\right)^{-1} \left(1 + \frac{z}{\bar{\zeta}_n}\right)^{-1}, \text{ and}$$

$$\varphi_q^-(z) = \frac{q}{q - \psi(z)} \frac{1}{\varphi_q^+(z)},$$

where $\bar{z}$ denotes the complex conjugate of z, and k is the least upper bound of the positive jumps.

In this case all solutions but one are complex, but we are given the asymptotics for the large solutions for a fairly broad class of processes, and the authors provide a practical numerical method for finding the smaller solutions. Excluded from the theorem above, but included in [73], is a statement concerning the approximate location of the solutions. The authors find that the majority of the solutions (all but a collection having an asymptotic density equal to 0) are complex numbers in $\mathcal{Q}_1$ with angle greater than $\pi/2 - \varepsilon$, where ε may be arbitrarily small.

3.2.4 Meromorphic processes

We recall from Chapter 2 that a meromorphic process is an "infinite version" of a hyper-exponential process which derives its name from the fact that its Laplace exponent,

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z^2 \sum_{n\geq 1} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} + z^2 \sum_{n\geq 1} \frac{a_n}{\rho_n(\rho_n - z)}, \tag{3.14}$$

may be continued to a real meromorphic function on $\mathbb{C}$. It is also important to remember the interlacing property (2.16) of the poles $\{\hat{\rho}_n\}_{n\geq 1}$, $\{\rho_n\}_{n\geq 1}$ of $\psi(z)$ and the solutions $\{-\hat{\zeta}_n\}_{n\geq 1}$, $\{\zeta_n\}_{n\geq 1}$ of the equation $\psi(z) = q$. We might expect that the Wiener-Hopf factorization for a meromorphic process can be obtained by simply extending the finite products in (3.5) to infinite ones. This intuition is correct, although the result is not trivial. In particular, it is not obvious that we can "factor" the series in (3.14) into a product representation like we

would for a rational function. See [66] for a proof that such a factorization is, in fact, possible and for a proof of the following key theorem. The proof of Theorem 10 follows Approach 1.

Theorem 10 (Theorem 1 in [66]). *Assume that $q > 0$. Then for $\mathrm{Re}(z) > 0$*

$$\varphi_q^+(-z) = \prod_{n\geq 1} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}}, \qquad \text{and} \qquad \varphi_q^-(z) = \prod_{n\geq 1} \frac{1+\frac{z}{\hat{\rho}_n}}{1+\frac{z}{\hat{\zeta}_n}}.$$

The distribution of $S_{\boldsymbol{e}(q)}$ can be identified as an infinite mixture of exponential distributions

$$\mathbb{P}(S_{\boldsymbol{e}(q)} \in \mathrm{d}x) = \beta_0 \delta_0(\mathrm{d}x) + \sum_{n\geq 1} \beta_n \zeta_n e^{-\zeta_n x} \mathrm{d}x,$$

where $\mathbb{P}(S_{\boldsymbol{e}(q)} \in \mathrm{d}x)$ is a measure supported on $\bar{\mathbb{R}}^+$, and the coefficients $\{\beta_n\}_{n\geq 0}$ are positive, satisfy $\sum_{n\geq 0} \beta_n = 1$, and can be computed as

$$\beta_0 = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\zeta_k}{\rho_k}, \qquad \text{and} \qquad \beta_n = \left(1-\frac{\zeta_n}{\rho_n}\right) \prod_{\substack{k\geq 1 \\ k\neq n}} \frac{1-\frac{\zeta_n}{\rho_k}}{1-\frac{\zeta_n}{\rho_n}}.$$

The distribution of $-I_{\boldsymbol{e}(q)}$ has the same form as above, with $\{\rho_n, \zeta_n\}_{n\geq 1}$ replaced by $\{\hat{\rho}_n, \hat{\zeta}_n\}_{n\geq 1}$.

Theorem 10 is especially useful when combined with the closed form formulas for the Laplace exponents of processes in the β and θ families and the interlacing property. We recall from Chapter 2 that for these families the poles are just functions of the index n that behave linearly or quadratically. Therefore, we may quickly and easily calculate truncated versions of the Wiener-Hopf factors and associated densities by numerically finding zeros of $q - \psi(z)$. The interlacing property tells us the zeros are real, and also gives us the approximate location at which to start our numerical search. Additionally, for large zeros we have asymptotic expansions for both families, detailed in [65, 66] and Chapter 5, which may be used as input for the numerical method. This technique is also useful in determining the density of $X_{\mathbf{e}(q)}$ which may be expressed in the following way:

Theorem 11 (Theorem 2 v. in [71]). *Let X be a meromorphic Lévy process which is not a compound Poisson process and let $q > 0$. Then,*

$$\mathbb{P}(X_{\boldsymbol{e}(q)} \in \mathrm{d}x) = q \left(\mathbb{I}(x>0) \sum_{n\geq 1} \frac{e^{-\zeta_n x}}{\psi'(\zeta_n)} - \mathbb{I}(x<0) \sum_{n\geq 1} \frac{e^{-\hat{\zeta}_n x}}{\psi'(\hat{\zeta}_n)} \right) \mathrm{d}x.$$

It is rare that we find processes for which we have explicit expressions for the density of X_t. In Theorem 11 we get the next best thing, that is, an expression for the density of $X_{\mathbf{e}(q)}$

which can be evaluated numerically very easily.

3.2.5 Processes with completely monotone jumps

We conclude this section with a brief overview of the Wiener-Hopf factorization for processes with completely monotone jumps although there are no known general formulas for this class. The motivation for this section is that it provides an opportunity to explain why many of the popular processes in mathematical finance, such as the VG, CGMY, and NIG processes do not have explicit Wiener-Hopf factorizations. It also allows us to mention an important theorem of Rogers [101] which demonstrates the connection between the Wiener-Hopf factors of processes with completely monotone jumps and random variables with exponential mixture distributions.

We recall that a VG process has Laplace exponent

$$\psi(z) = -c\log\left(1+\frac{z}{\hat{\rho}}\right) - c\log\left(1-\frac{z}{\rho}\right), \tag{3.15}$$

where the parameters are all positive. With this Laplace exponent, we have little hope of factoring the function $q/(q-\psi(z)$ into the product of Laplace transforms of infinitely divisible distributions. We also do not see a clear strategy for evaluating the integral of Approach 3. Therefore, our only hope lies with Approach 2. However, we see that $\psi(z)$ has branch points at $-\hat{\rho}$ and ρ and so it clearly cannot be extended to a meromorphic function on the left or right half-plane. We find this a common problem with the CGMY and NIG process as well: in both cases the Laplace exponents have branch points which cause difficulty.

There is some hope in our search for a formula for the Wiener-Hopf factors of completely monotone processes: we may characterize the distributions of $S_{\mathbf{e}(q)}$ and $I_{\mathbf{e}(q)}$ for all processes in this class. We say a positive (resp. negative) random variable, Z^+ (resp. Z^-), is a ***mixture of exponentials*** if we can express its characteristic function as

$$\mathbb{E}[e^{i\theta Z^+}] = \int_{(0,\infty]} \frac{x}{(x-i\theta)}\mu^+(\mathrm{d}x) \qquad \left(\text{resp. } \mathbb{E}[e^{i\theta Z^-}] = \int_{(0,\infty]} \frac{x}{(x+i\theta)}\mu^-(\mathrm{d}x)\right),$$

where $\mu^+(\mathrm{d}x)$ (resp. $\mu^-(\mathrm{d}x)$) is a probability distribution. We recall, for example, that the distribution of $S_{\mathbf{e}(q)}$ for hyper-exponential processes with positive Gaussian component is a finite mixture of exponentials. The following theorem builds the connection between completely monotone processes and mixtures of exponentials.

Theorem 12 (Theorem 2 in [101]). *Let X be a Lévy process, and let $S_{e(q)}$, $I_{e(q)}$ be defined with respect to X.*

1. *If X is a completely monotone process then for each $q > 0$ the random variables $S_{e(q)}$ and $I_{e(q)}$ are mixtures of exponentials.*

2. *If for some $q > 0$, the random variables $S_{e(q)}$ and $I_{e(q)}$ are mixtures of exponentials, then X is a completely monotone process.*

What we can gather from this theorem is that it might be possible to find a good approximation of the distributions of $I_{\mathbf{e}(q)}$ and $S_{\mathbf{e}(q)}$ by some finite mixtures of exponentials. In Chapter 8 we demonstrate precisely how this can be done, by approximating completely monotone processes with hyper-exponential processes.

3.3 Two examples of applications in finance

Problems in finance and actuarial science are very often concerned with so-called exit problems. By this we mean problems that deal with the exit of the process from an interval. It seems natural that we should be able to formulate such exit problems in terms of extrema processes I and S, and luckily, with the help of the Wiener-Hopf factorization, this is often the case. Further, once we have such a formulation, knowledge of the Wiener-Hopf factors usually provides us with some method of solution. In this section we present two examples which demonstrate the usefulness of the Wiener-Hopf factorization in pricing options.

We begin by showing how the Wiener-Hopf factorization may be used to price barrier options, which is the primary application for the theoretical results developed in Chapter 8. Then we show its usefulness in pricing perpetual American options. This final example is included primarily to demonstrate the Wiener-Hopf factorization's versatility: we show two different methods of obtaining the price via the Wiener-Hopf factorization. Also, it allows us introduce two other interesting fluctuation identities.

The reader will recall that $A = \{A_t : t \geq 0\}$ represents our stock price process which is based on an underlying Lévy process X (see definition in Section 2.1.4), τ_x^+ is the first strict passage time for the process X at level x (see again Section 2.1.4), and $r > 0$ is the discount rate.

Example: Down and out barrier options

We wish to value the option to sell a stock at some expiry time $T > 0$ for strike price $K > 0$ on the condition that the option is invalid if the stock price falls below some barrier $0 < B < A_0$ prior to time T. This is known as a down-and-out barrier put option, and in mathematical terms the quantity we are interested in calculating is

$$D(A_0, K, B, T) := e^{-rT}\mathbb{E}\left[(K - A_T)^+ \mathbb{I}\left(\inf_{0\leq t\leq T} A_t > B\right)\right].$$

By factoring out the constant A_0 and dropping the discounting term, we can instead solve the equivalent problem of determining

$$f(T) := \mathbb{E}[(k - e^{X_T})^+ \mathbb{I}(I_T > b)],$$

where $k := K/A_0$ and $b := \log(B/A_0)$. Now, if we take the Laplace transform of $f(t)$ we may replace the deterministic time T by the independent (of X) random time $\mathbf{e}(q)$. That is, we define the function $F(q)$ as

$$F(q) := \int_{\mathbb{R}^+} qe^{-qt} f(t)\mathrm{d}t = \mathbb{E}[(k - e^{X_{\mathbf{e}(q)}})^+ \mathbb{I}(I_{\mathbf{e}(q)} > b)].$$

Accordingly, we can solve our problem if we can determine $F(q)$ and then invert the Laplace transform to recover $f(t)$. We notice, however, that due to the Wiener-Hopf factorization we may write,

$$F(q) = \mathbb{E}[(k - e^{S_{\mathbf{e}(q)} + I_{\mathbf{e}(q)}})^+ \mathbb{I}(I_{\mathbf{e}(q)} > -b)],$$

and so we have reformulated the problem in terms of the random variables $S_{\mathbf{e}(q)}$ and $I_{\mathbf{e}(q)}$. The approach we have just described is originally due to Jeannin and Pistorius [60].

If we now suppose that we can determine the Wiener-Hopf factors, and identify the distributions of these random variables – perhaps even their densities – then the problem becomes tractable. For example, we might consider the scenario where X is a hyper-exponential process with positive Gaussian component. In this case we have simple densities for $S_{\mathbf{e}(q)}$ and $I_{\mathbf{e}(q)}$ and we may develop an explicit expression for $F(q)$ in terms of the now familiar zeros and poles.

Example: Perpetual American options and the Pecherskii-Rogozin identity

The perpetual American put gives the buyer the option to sell a stock at any point in the future for a strike price $K > 0$. The value of such an option, in mathematical terms, is given by

$$v(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - A_\tau)^+],$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$, and where we change the definition of A_0 slightly to depend on x, i.e. $A_0(x) := \exp(x)$. It turns out that there exists an optimal stopping time which maximizes v, which we will denote τ^*. This stopping time corresponds to the strict first passage time of the process X below a barrier x^*. In other words, we can reformulate the problem as an exit problem as follows:

Theorem 13 (Theorem 3 in [4])**.** *Assume $r > 0$ or $r = 0$ and our process X obeys $\lim_{t\to\infty} X_t = \infty$ almost surely, then*

$$v(x) = \frac{\mathbb{E}[(K\mathbb{E}[e^{I_{\mathbf{e}(r)}}] - e^{x+I_{\mathbf{e}(r)}})^+]}{\mathbb{E}[e^{I_{\mathbf{e}(r)}}]},$$

with optimal stopping time

$$\tau^* = \inf\{t \geq 0 : X_t < x^*\},$$

and barrier,

$$x^* = \log(K)\mathbb{E}[e^{I_{\mathbf{e}(r)}}].$$

Here $\mathbf{e}(r)$ plays the role of $\mathbf{e}(q)$ and the reader should remember that we follow the convention that $\mathbf{e}(r) = \infty$ almost surely when $r = 0$. We see, as with the example for barrier options, that if we can identify the distribution of the random variable $I_{\mathbf{e}(r)}$ then we have a chance to determine the optimal barrier, the optimal stopping time, and the value of our option.

This problem also has another connection to the Wiener-Hopf factorization that involves the well-known Pecherskii-Rogozin identity, which was first introduced in [95]. For $w > 0$ and r, $z \geq 0$, the Pecherskii-Rogozin identity relates the double Laplace transform of the

strict first passage time and overshoot with the Wiener-Hopf factors as follows:

$$\int_{\mathbb{R}^+} e^{-wx}\mathbb{E}[e^{-q\tau_x^+ - z(X_{\tau_x^+}-x)}]\mathrm{d}x = \frac{1}{w-z}\left(1 - \frac{\varphi_r^+(-w)}{\varphi_r^+(-z)}\right). \tag{3.16}$$

Using the Wiener-Hopf factorization, together with the fluctuation identity

$$\mathbb{E}[e^{-r\tau_x^+ - z(X_{\tau_x^+}-x)}\mathbb{I}(\tau_x^+ < \infty)] = \frac{\mathbb{E}[e^{-zS_{\mathbf{e}(r)}}\mathbb{I}(S_{\mathbf{e}(r)} > x])}{\mathbb{E}[e^{-zS_{\mathbf{e}(r)}}]} \tag{3.17}$$

it is quite easy to obtain (3.16) (see [4] for details and a proof of (3.17)). The connection with the problem of the American put, is that for the class of regular Lévy processes of exponential type (first introduced in [21]) which includes the popular NIG and CGMY ($Y \in (0,2)$) processes, we can use (3.16) to obtain the Fourier transform of $v(x)$,

$$\int_{\mathbb{R}} e^{i\lambda x}v(x)\mathrm{d}x = K\frac{e^{i\lambda x^*}}{i\lambda(i\lambda+1)}\phi_r^-(-\lambda), \tag{3.18}$$

which is valid for $\lambda \in \mathbb{R}$. For the details of the proof of (3.18) using the Pecherskii-Rogozin identity see [4], and for the original derivation see Section 4.2 in [21]. This gives us another method to evaluate $v(x)$ provided we are able to determine the negative Wiener-Hopf factor and successfully perform a (numerical) Fourier inversion.

Chapter 4

The exponential functional

In this chapter we introduce the exponential functional of a Lévy process and its primary financial application, the Asian option. We state and prove a verification result for determining the distribution of the exponential functional involving the Mellin transform. This result has been successfully applied in both cases for which the distribution of the exponential functional of a process with two-sided jumps is known. We demonstrate, with an example, how the verification result may be applied in the case of processes with jumps of rational transform and give the distribution of the exponential functional for such processes; this gives an overview of the recent work of Kuznetsov [69] and Cai and Kou [27]. Additionally, we examine the connection, via the Lamperti transform, between exponential functionals and positive self-similar Markov processes. We review an interesting example from [72] that demonstrates how to derive the density of the supremum of a stable process from the density of the exponential functional of a hypergeometric process. In a related example, we show how to derive the distribution of an important homogeneous functional related to stable processes.

4.1 Introduction

Exponential functionals are important and useful objects, not only in mathematical finance, but also in many other areas of probability theory. They play a role in such fields as self-similar Markov processes, random processes in random environments, fragmentation processes, and branching processes. They are also connected with generalized Ornstein-Uhlenbeck processes where their distribution appears as the stationary measure. A good survey on the topic, covering both theory and applications, can be found in [15].

In this chapter we will highlight some recent findings regarding the distribution of the exponential functional. We describe two techniques–one involving the Mellin transform (Section 4.2) and the other involving the Lamperti transform (Section 4.3)–which have been successfully applied to determine the distribution of the exponential functional. The former technique is the most important for this work; we apply it again in Chapter 6 to derive the distribution of the exponential functional of a meromorphic process. Although the result for meromorphic processes generalizes an existing result for hypergeometric processes (see definition in Section 4.3.3), the example of the hypergeometric process is still interesting. This is because of its connection, through the Lamperti transform, with the density of the supremum of a stable process (this density is our focus in Chapter 7) and with the distribution of the homogeneous functional $\mathcal{A}_r$ (see Example 3 in Section 4.3.3 and Corollary 1). We explore the connection between hypergeometric processes and stable processes in Section 4.3.3. We also show, with various examples throughout the chapter, the role of the exponential functional in pricing Asian options. In the remainder of this Introduction, we define the exponential functional and discuss its role in pricing Asian options.

4.1.1 The exponential functional

We define the ***exponential functional*** of a Lévy process X to be the random variable

$$I_\zeta(X) := \int_0^\zeta e^{X_t} \mathrm{d}t,$$

where ζ is either $\mathbf{e}(q)$ or ∞. We will not use the notation $I_\zeta(X)$ directly, rather we will write $I_q(X)$ if $\zeta = \mathbf{e}(q)$ and $I_\infty(X)$ if $\zeta = \infty$ (equivalently $\zeta = \mathbf{e}(q)$ and $q = 0$). In this latter case, we know from Theorem 1 in [15] that the random variable $I_\infty(X)$ is well defined provided X drifts to $-\infty$. By a slight abuse of nomenclature, we will also refer to

$$I_u(X) := \int_0^u e^{X_t} \mathrm{d}t$$

as the exponential functional of X, where u is a finite deterministic time. The reader should note that $I_u(X)$ is often referred to as the *additive functional* in the literature; in this instance our nomenclature is non-standard. When there is no danger of ambiguity, we will drop the reference to the process X in the notation and write simply I_q, I_∞, and I_u.

Determining the distribution of the exponential functional can be quite complicated, and the number of explicit results is small. For process with either no jumps, or one-sided jumps, the distribution is known explicitly for the case when X is either: a Poisson process,

a scaled Brownian motion with drift (see Section 4.3.3), or one of several classes of spectrally one-sided processes (see for example [31, 79, 89, 91]). However, for processes with two-sided jumps, we have explicit distributions only for processes with jumps of rational transform, and for hypergeometric processes [72]. In the latter case, the processes in question may have both infinite activity and infinite variation. To date, this is the only example of a two-sided, infinite activity/variation process for which we can identify the distribution of the exponential functional. Even this result is not completely general: our knowledge is limited to the distribution of I_q for only one value of q which depends on the parameters of the process.

We close this section by presenting the primary financial application of the exponential functional, the Asian option. We will refer to this example throughout the chapter and in Chapters 5 and 6.

Example: Asian Options

The exponential functional is clearly visible in the pricing formula for Asian options and it will be the key in our approach to calculating option prices. We recall from Section 2.1.4 that the stock price, A, based on a Lévy process X, is defined by $A_t := A_0 \exp(X_t)$, and we assume that our measure $\mathbb{P}$ is risk neutral. We are interested in calculating the price of an arithmetic, continuously monitored, fixed strike Asian call option, which is given by

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right], \tag{4.1}$$

where T, $K > 0$ are the expiry time and strike price respectively. Finding an explicit expression for $C(A_0, K, T)$ is a difficult problem, and, even for the simple case of Brownian motion it remains open. The difficulty arises from the fact that the process Z, defined by $Z_t = I_t$, $t \geq 0$, is not a Markov process (see Section 4.3.1). In other words, the value of the option depends on the entire path of Z over the time interval $[0, T]$ rather than the value of Z solely at time T. Our best hope for calculating $C(A_0, K, T)$ is to find an implicit, or semi-explicit formula which is easy to work with from a numerical perspective.

By factoring out the constants $1/T$ and A_0, we get $C(A_0, K, T) = \exp(-rT) \times A_0/T$ $\times f_a(TK/A_0, T)$ where

$$f_a(k, t) := \mathbb{E}\left[\left(\int_0^t e^{X_u}\mathrm{d}u - k\right)^+\right]. \tag{4.2}$$

We see that determining $f_a(k,t)$ is equivalent to obtaining a price. Proceeding as we did in Section 3.3 with barrier options, we replace the deterministic time t with the random time $\mathbf{e}(q)$ by taking the Laplace transform

$$h_a(k,q) := q\int_{\mathbb{R}^+} e^{-qt} f_a(k,t)\mathrm{d}t = \mathbb{E}[f_a(k,\mathbf{e}(q))] = \mathbb{E}[(I_q(X)-k)^+]. \tag{4.3}$$

We will see in later examples that the function $h_a(k,q)$ is used either directly or indirectly to determine $f_a(k,t)$ and subsequently the price. If the distribution of $I_q(X)$ is tractable enough, then we have a good chance of deriving an explicit expression for $h_a(k,q)$. This turns out to be the case when X is a scaled Brownian motion with drift (see Section 4.3.3). However, when X is a process with jumps, the distribution of $I_q(X)$ will be too complicated for a direct approach. We show in Section 4.2 that our method may still be successful, provided that we can find a nice expression for the Mellin transform of $h_a(k,q)$.

4.2 The Mellin transform

In this section we will study the Mellin transform of the random variable I_q which we define as the function

$$\mathcal{M}(I_q,z) := \mathbb{E}[I_q^{z-1}], \quad z \in 1+i\mathbb{R}.$$

We plan to use the Mellin transform to determine the distribution of the exponential functional. To do this, will use a verification result based on a functional equation involving the Mellin transform and the Laplace exponent of the underlying process. We will see that functions satisfying the functional equation, and some growth constraints, can be identified as Mellin transforms of exponential functionals. Although finding the correct candidate function for the verification result may be challenging, this approach has been used successfully to determine the distribution of the exponential functional for processes with jumps of rational transform [69], for hypergeometric processes [72], and in Chapter 6 we use it again for meromorphic processes.

In Section 4.2.2 we will give a proof of the verification result, and in Section 4.2.3 we will use it to derive the distribution of $I_q(X)$ when X is a Lévy process with jumps of rational transform. In particular, for the special case where X is a hyper-exponential process, we will show that we can express $I_q(X)$ in terms of products of well-known random variables. In Section 4.2.1 we give precise statements of these results, which were originally derived by

Cai and Kou [27] and Kuznetsov [69], and remind the reader of the necessary facts about processes with jumps of rational transform. The results and proof for hyper-exponential processes can be viewed as inspiration for the results in Chapter 6 on meromorphic processes. To conclude our discussion on the Mellin transform, we show in Section 4.2.4 how it may be used to solve the pricing problem for Asian options.

4.2.1 The distribution of I_q for processes with jumps of rational transform

Before stating any theorems regarding processes with jumps of rational transform, we must recall some facts about their Laplace exponents $\psi(z)$, and the solutions of the equation $\psi(z) = q$ (see equation (2.9) and Theorem 2 for full details). A process with jumps of rational transform is a finite activity process, and therefore necessarily a compound Poisson process. As usual we will denote the parameter controlling the intensity of the jumps by λ. The Laplace exponent $\psi(z)$ has N poles at points $\{\rho_n\}_{1\leq n\leq N}$ located in the half-plane $\mathrm{Re}(z) > 0$. Of these only ρ_1 is required to be real, and we assume that the poles are ordered by their real component, i.e. $\rho_1 < \mathrm{Re}(\rho_2) < \ldots < \mathrm{Re}(\rho_N)$. Pole ρ_i has multiplicity M_i so that the pole count–with multiplicity–in the positive half-plane is given by $P = \sum_{1\leq n\leq N} M_n$. The poles $\{-\hat{\rho}_n\}_{1\leq n\leq \hat{N}}$ in the negative half-plane can be defined in an analogous way, with $-\hat{\rho}_i$, and $\hat{M}_i$ denoting the ith pole and its multiplicity, and $\hat{N}$ and $\hat{P}$ the pole count and pole count with multiplicity respectively. The total pole count, with multiplicity, is denoted $R = P + \hat{P}$.

The equation $\psi(z) = q$ has K solutions in the positive half-plane, $\hat{K}$ solutions in the negative half-plane, and $Q = K + \hat{K}$ total solutions [1]. The solutions in the positive (resp. negative) half-plane are denoted $\{\zeta_n\}_{1\leq n\leq N}$ (resp. $\{-\hat{\zeta}_n\}_{1\leq n\leq \hat{N}}$) and we presume they are ordered by their real component like the poles. Further, the values P and K, and $\hat{P}$ and $\hat{K}$, are related in a manner that depends on the values of the Gaussian component σ and the drift a of the process:

$$(K, \hat{K}) = \begin{cases} (P+1, \hat{P}+1) & \sigma > 0 \\ (P+1, \hat{P}) & \sigma = 0 \text{ and } a > 0 \\ (P, \hat{P}+1) & \sigma = 0 \text{ and } a < 0 \\ (P, \hat{P}) & \sigma = 0 \text{ and } a = 0 \end{cases}.$$

[1] When we count solutions of the equation $\psi(z) = q$ we always assume that our count also tracks multiplicity.

Finally, the solutions $-\hat{\zeta}_1$, and ζ_1 are real and satisfy

$$\hat{\zeta}_1 < \hat{\rho}_1, \quad \text{and} \quad \zeta_1 < \rho_1. \tag{4.4}$$

We recall that hyper-exponential processes are just processes with jumps of rational transform for which all of the poles are simple and real valued. In this case, all solutions of $\psi(z) = q$ are also simple and real valued, and we may extend (4.4) to the interlacing property:

$$\begin{aligned} 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 \dots \\ 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 \dots . \end{aligned} \tag{4.5}$$

We are now able to state the first result, which concerns hyper-exponential processes. The following theorem was proven by Cai and Kou [27] for the case where the Gaussian component of the process is non zero. A generalization of their results for processes with jumps of rational transform was published by Kuznetsov [69] in 2012. In this paper he also proved the remaining case ($\sigma = 0$) for hyper-exponential processes. In the theorem, $G_{(\alpha,\beta)}$ stands for a gamma random variable with shape and scale parameters α and β respectively, and $B_{(\alpha,\beta)}$ stands for a beta random variable, again with parameters α and β. Further, we follow the convention that the empty product $\prod_{k=1}^{0}$ is equal to 1.

Theorem 14 (Theorem 4.3 in [27] and Theorem 1 in [69])**.** *Let X be a hyper-exponential Lévy process, and assume that $q > 0$ or $q = 0$ and $\mathbb{E}[X_1] < 0$, then*

$$I_q(X) \overset{d}{=} A B_{(1,\hat{\zeta}_1)} \frac{\prod_{k=1}^{\hat{K}-1} B_{(\hat{\rho}_k+1, \hat{\zeta}_{k+1}-\hat{\rho}_k)}}{\prod_{k=1}^{P} B_{(\zeta_k, \rho_k-\zeta_k)}}, \tag{4.6}$$

where,

$$A := \begin{cases} \dfrac{2}{\sigma^2 G_{(\zeta_K,1)}} & \sigma > 0 \\ \dfrac{G_{\left(\hat{\rho}_{\hat{K}}+1,1\right)}}{|a| G_{(\zeta_K,1)}} & \sigma = 0 \text{ and } a > 0 \\ \dfrac{1}{|a|} & \sigma = 0 \text{ and } a < 0 \\ \dfrac{G_{\left(\hat{\rho}_{\hat{K}}+1,1\right)}}{q+\lambda} & \sigma = a = 0 \end{cases},$$

and where (4.6) is a product of independent random variables.

The main contribution of [69] is that we may find an expression for the density of $I_q(X)$ in the general setting, where X is a process with jumps of rational transform. The density is expressed in terms of the function $\mathcal{G}(z)$, which we define for a process with jumps of rational transform as

$$\mathcal{G}(z) := \frac{\prod_{k=1}^{K} \Gamma(\zeta_k - z + 1)}{\prod_{n=1}^{N} \Gamma(\rho_n - z + 1)^{M_n}} \times \frac{\prod_{n=1}^{\hat{N}} \Gamma(\hat{\rho}_n + z)^{M_n}}{\prod_{k=1}^{\hat{K}} \Gamma(\hat{\zeta}_k + z)}, \tag{4.7}$$

and in terms of the Meijer-G function, $\mathbf{G}_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right)$ (See Appendix A).

For any $z \in \mathbb{C}$, and $n \in \mathbb{N}$, we define the vector $[z]_n \in \mathbb{C}^n$ as

$$[z]_n = (z, z, \ldots, z).$$

Then, we define the vectors $\boldsymbol{\rho} \in \mathbb{C}^{R+1}$ and $\boldsymbol{\zeta} \in \mathbb{C}^{Q}$ as

$$\boldsymbol{\rho} = (1, [1 - \hat{\rho}_1]_{\hat{M}_1}, [1 - \hat{\rho}_2]_{\hat{M}_2}, \ldots [1 - \hat{\rho}_{\hat{N}}]_{\hat{M}_{\hat{N}}}, [1 + \rho_1]_{M_1}, [1 + \rho_2]_{M_2}, \ldots, [1 + \rho_N]_{M_N}), \tag{4.8}$$

$$\boldsymbol{\zeta} = (1 + \zeta_1, 1 + \zeta_2, \ldots, 1 + \zeta_K, 1 - \hat{\zeta}_1, 1 - \hat{\zeta}_2, \ldots, 1 - \hat{\zeta}_{\hat{K}}). \tag{4.9}$$

Under the assumption that $\Gamma(z)\mathcal{G}(z)$ has only simple poles [2] we have the following theorem:

Theorem 15 (Proposition 3 in [69]). *Let X be a Lévy process with jumps of rational transform, and assume that $q > 0$ or $q = 0$ and $\mathbb{E}[X_1] < 0$, then the density $p(x)$ of the exponential functional $I_q(X)$ satisfies*

$$p(x) = \frac{A}{\mathcal{G}(1)} \times \mathbf{G}_{R+1,Q}^{K,\hat{P}+1}\left((Ax)^{-1} \,\middle|\, \begin{matrix} \boldsymbol{\rho} \\ \boldsymbol{\zeta} \end{matrix} \right),$$

where,

$$A := \begin{cases} \frac{\sigma^2}{2} & \sigma > 0 \\ |a| & \sigma = 0 \text{ and } a \neq 0 \\ q + \lambda & \sigma = a = 0 \end{cases}.$$

A few comments on Theorem 15 are in order. First, from the series formulation of the Meijer-G function, we get series representations and asymptotic expansions for $p(x)$ (see Proposition 3 (i),(ii), and (iii) in [69]). Second, our assumption that $\Gamma(z)\mathcal{G}(z)$ has only simple poles is

[2] See Assumption A in [69] for sufficient conditions in terms of the poles $\{-\hat{\rho}_n\}$, solutions $\{-\zeta_n\}$, and function $q - \psi(z)$, which together ensure this assumption is satisfied

purely for simplification. We may also obtain densities when this assumption fails, but the resulting expression is more complicated. As already stated, the key to proving Theorem 14 and Theorem 15 is finding an expression for the Mellin transform of $I_q(X)$. In the next section, we discuss a verification result which helps us derive this expression.

4.2.2 A verification result

The proof of the verification result is relatively short, so we give a sketch of the details below. We will do so in two steps. First, we will prove a lemma due to Maulik and Zwart [85] and Carmona et. al. [28] showing that the Mellin transform satisfies a functional equation involving the Laplace exponent of the process. Second, we will give a sketch of a proof of the converse claim, due to Kuznetsov and Pardo [72], that any function satisfying the functional equation (plus some other technical conditions) must be the Mellin transform.

The following useful lemma is due to Maulik and Zwart [85] for the case $q = 0$ and Carmona et. al. [28] for the case $q > 0$. The proof we present below is a combination of the proofs from these two papers.

Lemma 1 (Proposition 3.1 in [28] and Lemma 2.1 in [85])**.** *Let $q \geq 0$ and X be a Lévy process with Laplace exponent $\psi(z)$. If $z > 0$ and $q - \psi(z) > 0$, we have*

$$\mathcal{M}(I_q, z+1) = \frac{z}{q - \psi(z)} \mathcal{M}(I_q, z), \tag{4.10}$$

where the equality is interpreted to mean that both sides can be infinite.

Proof. We may treat the cases $q = 0$ and $q > 0$ identically except for a final step. The common first piece of the proof goes as follows: we integrate the following identity

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(I_t - I_u\right)^z = -z(I_t - I_u)^{z-1} e^{X_u},$$

over the interval $[0, t]$ to obtain

$$I_t^z = z \int_0^t \left(I_t - I_u\right)^{z-1} e^{X_u} \mathrm{d}u. \tag{4.11}$$

Now, we observe that

$$I_t - I_u = e^{X_u} \int_0^{t-u} e^{(X_{u+s} - X_u)} \mathrm{d}s \stackrel{d}{=} e^{X_u} I_{t-u}, \tag{4.12}$$

where the last equality follows from the fact that the process $\tilde{X}$, defined by $\tilde{X}_s := X_{t+s} - X_t$, is independent of the process X up until time t, and has the same distribution as X (see Theorem 17). Plugging (4.12) into (4.11) and taking expectations, gives, after an application of Fubini's theorem,

$$\mathbb{E}[I_t^z] = z \int_0^t e^{u\psi(z)} \mathbb{E}[I_{t-u}^{z-1}] \mathrm{d}u. \tag{4.13}$$

For the case $q = 0$, we make the change of variables $u \mapsto t - v$ in (4.13), and calculate the limit as $t \to \infty$ via an application of l'Hôpital's rule:

$$\mathbb{E}[I_\infty^z] = -z \lim_{t\to\infty} \frac{e^{-t\psi(z)}\mathbb{E}[I_t^{z-1}]}{\psi(z)e^{-t\psi(z)}} \mathrm{d}u = -\frac{z}{\psi(z)} \mathbb{E}[I_\infty^{z-1}].$$

In this final step we use the assumption that $\psi(z) < 0$ to justify our use of l'Hôpital's rule. When $q > 0$ we plug (4.13) into the following identity:

$$\mathbb{E}[I_q^z] = q \int_{\mathbb{R}^+} e^{-qt} \mathbb{E}[I_t^z] \mathrm{d}t.$$

An application of Fubini's theorem with the same change of variables, and the fact that $q - \psi(z) > 0$ yields the result. □

Now, we state and prove the verification result. Although the statement of this theorem and the associated proof originally appeared in [72] we will use here the statement and abbreviated proof from [69].

Theorem 16 (Proposition 2 in [69])**.** *Assume that Cramér's condition is satisfied: there exists $z_0 > 0$ such that the Laplace exponent $\psi(z)$ is finite for all $z \in (0, z_0)$ and $\psi(\theta) = q$ for some $\theta \in (0, z_0)$. If a function $f(z)$ satisfies the following three properties:*

(i) $f(z)$ is analytic and zero-free in the strip $\mathrm{Re}(z) \in (0, 1 + \theta)$;

(ii) $f(1) = 1$ and $f(z + 1) = zf(z)/(q - \psi(z))$ for all $z \in (0, \theta)$; and

(iii) $|f(z)|^{-1} = o(\exp(2\pi|\mathrm{Im}(z)|))$ as $\mathrm{Im}(z) \to \infty$, uniformly in $\mathrm{Re}(z) \in (0, 1 + \theta)$,

then $\mathcal{M}(I_q, z) \equiv f(z)$ for $\mathrm{Re}(z) \in (0, 1 + \theta)$.

Proof (sketch). First, we gather some facts about $\mathcal{M}(I_q, z)$. From Cramér's condition and Lemma 2 in [100] we know that $\mathcal{M}(I_q, z)$ can be extended to an analytic function in the vertical strip $\mathrm{Re}(z) \in (0, 1 + \theta)$. Further, since $|\mathcal{M}(I_q, z)| < \mathcal{M}(I_q, \mathrm{Re}(z))$ we see that $\mathcal{M}(I_q, z)$

is bounded on the strip $[\theta/2, 1+\theta/2]$. Lastly, invoking once again Cramér's condition, we see that the sufficient conditions of Lemma 1 are satisfied on the interval $(0, \theta)$, and so $\mathcal{M}(I_q, z)$ satisfies the functional equation in (ii).

This last point ensures that the function $F(z) = \mathcal{M}(I_q, z)/f(z)$ is a periodic function with period 1. Due to condition (i), $F(z)$ may be extended to an analytic function in the entire complex plane, and condition (iii) and the boundedness of $\mathcal{M}(I_q, z)$ imply that $F(z) = o(\exp(2\pi|\mathrm{Im}(z)|))$ uniformly in $\mathrm{Re}(z) \in \mathbb{R}$. Any function which is analytic, periodic with period equal to one, and which satisfies this asymptotic condition must be identically equal to a constant (see proof of Proposition 2 in [72]). Since $F(1) = 1$ we conclude that $F(z) \equiv 1$, that is $\mathcal{M}(I_q, z) \equiv f(z)$. □

To apply the verification result of Theorem 16 we see that we need a candidate function $f(z)$ that satisfies the three criteria. In the following section, we will see how to construct this function for a process with jumps of rational transform. We will use similar ideas in Chapter 6 for meromorphic processes.

4.2.3 An application of the verification result for processes with jumps of rational transform

Theorems 14 and 15 are proven in [69] using the verification result with the following candidate function:

$$h(z) = A^{1-z} \times \Gamma(z) \times \frac{\mathcal{G}(z)}{\mathcal{G}(1)}. \tag{4.14}$$

Here, A is the constant in Theorem 15, and the function $\mathcal{G}(z)$ is defined in (4.7). It is easy to check that $h(z)$ satisfies the criteria of the verification result of Theorem 16, but what is the origin of $h(z)$? How do we deduce the proper form for the candidate function?

We give a nearly complete answer to these questions by using a simple process as an example. The extension to a general hyper-exponential process, or a general process with jumps of rational transform is straight-forward and is discussed in Theorem 1 in [69]. Accordingly, let X be the process with Laplace exponent

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z\frac{a_1}{\rho_1 - z} - z\left(\frac{\hat{a}_1}{\hat{\rho}_1 + z} + \frac{\hat{a}_2}{\hat{\rho}_2 + z}\right), \tag{4.15}$$

where $\hat{a}_1$, $\hat{a}_2$, a_1, $\hat{\rho}_1$, $\mathrm{Re}(\hat{\rho}_2)$, ρ_1, $\sigma > 0$. Note that we have not specified that the pole $-\hat{\rho}_2$ is real; if it is real, then X is a hyper-exponential process. In either case, the function $\psi(z)$ has only three poles $-\hat{\rho}_1$, $-\hat{\rho}_2$, and ρ_1, and since $\sigma > 0$, the equation $\psi(z) = q$ has exactly five solutions, $-\hat{\zeta}_3, -\hat{\zeta}_2, -\hat{\zeta}_1, \zeta_1$, and ζ_2. For the remainder of this section, we assume the functions $h(z)$ and $\mathcal{G}(z)$ are defined with respect to X.

Our approach is to solve the functional equation of Theorem 16 (ii)

$$f(z+1) = \frac{z}{q - \psi(z)} f(z), \tag{4.16}$$

and then verify (or ensure) that one of our solutions satisfies the remaining requirements of the verification result. In deriving a solution, we will take advantage of the fact that $\psi(z)$ is a rational function so that we may write

$$\frac{z}{q - \psi(z)} = C \frac{z(z+\hat{\rho}_1)(z+\hat{\rho}_2)(\rho_1 - z)}{(z+\hat{\zeta}_3)(z+\hat{\zeta}_2)(z+\hat{\zeta}_1)(\zeta_1 - z)(\zeta_2 - z)}, \tag{4.17}$$

where C is a constant. It is easy to verify that $C = 2/\sigma^2$ from the fact that $(q - \psi(z))$ $/(-\sigma^2 z^2/2) \to 1$ as $z \to +\infty$.

The functional equation (4.16) is reminiscent of the recursion formula for the gamma function

$$\Gamma(z+1) = z\Gamma(z), \tag{4.18}$$

and this is precisely what we will use to find a solution. Let us consider each factor of (4.17) separately and solve simpler functional equations of the type

$$f^+(z+1) = (z+a)^k f^+(z), \quad f^-(z+1) = (a-z)^k f^-(z), \quad \text{and} \quad f^c(z+1) = \frac{2}{\sigma^2} f^c(z),$$

where $a \in \{0, \hat{\rho}_1, \hat{\rho}_2, \rho_1, \hat{\zeta}_3, \hat{\zeta}_2, \hat{\zeta}_1, \zeta_1, \zeta_2\}$, and $k \in \{-1, 1\}$. The first two equations may be readily solved using (4.18). This approach yields solutions of the form $f^+(z) = \Gamma(z+a)^k$, and $f^-(z) = \Gamma(a - z + 1)^{-k}$. The final equation may be solved by inspection, and has solution $f^c(z) = (\sigma^2/2)^{1-z}$. These facts, together with property (4.4) and our knowledge of

the domain of the gamma function, demonstrate that the function

$$g(z) = \left(\frac{\sigma^2}{2}\right)^{1-z} \times \Gamma(z) \times \frac{\Gamma(\zeta_1 - z + 1)\Gamma(\zeta_2 - z + 1)}{\Gamma(\rho_1 - z + 1)} \times \frac{\Gamma(z + \hat{\rho}_1)\Gamma(z + \hat{\rho}_2)}{\Gamma(z + \hat{\zeta}_1)\Gamma(z + \hat{\zeta}_2)\Gamma(z + \hat{\zeta}_3)} \tag{4.19}$$

solves (4.16) for $\mathrm{Re}(z) \in (0, \zeta_1)$ and is analytic and zero free for $\mathrm{Re}(z) \in (0, 1 + \zeta_1)$. Further, $g(1) = \mathcal{G}(1)$, which shows that the function $h(z) = g(z)/\mathcal{G}(1)$ satisfies $h(1) = 1$. We observe that $h(z)$ is exactly the candidate function of (4.14).

Now that we have some insight into the origin of the candidate function, let us continue by verifying the remainder of the criteria of Theorem 16. In other words, let us demonstrate that $h(z)$ is the Mellin transform of the exponential functional. What remains is to check whether X satisfies Cramér's condition, and whether the asymptotic condition in (iii) holds. Property (4.4), and the definitions of ρ_1 and ζ_1, show that Cramér's condition is satisfied for $z_0 = \rho_1$ and $\theta = \zeta_1$. We verify the asymptotic condition only for the case $\hat{\rho}_2 \in \mathbb{R}$, in other words, when X is a hyper-exponential process; the method of verification in the general case relies on the same reasoning. To proceed we will use a well known property of the gamma function, specifically, we use Formula 8.328.1 in [49] which states

$$\lim_{|y| \to \infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2} - x} = \sqrt{2\pi}, \quad x, y \in \mathbb{R}. \tag{4.20}$$

It is known that the limit exists uniformly in x on compact subsets of $\mathbb{R}$ (as can be seen from Stirling's asymptotic formula for the gamma function). Formula (4.20) ensures that we may write $|h(z)|^{-1}$ as

$$|h(z)|^{-1} = \mathcal{G}(1) \times \left(\frac{\sigma^2}{2}\right)^{z-1} \times e^{\frac{\pi}{2}|\mathrm{Im}(z)|} |\mathrm{Im}(z)|^c \times \gamma(z),$$

where c depends on $\mathrm{Re}(z)$, and $\gamma(z) \to (\sqrt{2\pi})^{-1}$ as $|\mathrm{Im}(z)| \to \infty$. This shows that $|h(z)|^{-1} = o(\exp(2\pi|\mathrm{Im}(z)|))$ as $|\mathrm{Im}(z)| \to \infty$, and therefore, that $h(z) \equiv \mathcal{M}(z)$ for $\mathrm{Re}(z) \in (0, 1 + \zeta_1)$.

To conclude this section, we show how to use the Mellin transform to prove the results of Theorem 14 and 15 in the context of our present example. First, we recall that the Mellin

transforms of a gamma random variable $G_{(\alpha,\beta)}$ and beta random variable $B_{(\alpha,\beta)}$ are given by

$$\mathcal{M}(G,z) = \beta^{1-z}\frac{\Gamma(z+\alpha-1)}{\Gamma(\alpha)}, \quad z+\alpha-1>0, \text{ and}$$
$$\mathcal{M}(B,z) = \frac{\Gamma(\alpha+z-1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+z-1)}, \quad z+\alpha-1>0$$

respectively. If $\hat{\rho}_2 \in \mathbb{R}$, then X is a hyper-exponential process. This means that the poles of $\psi(z)$, and zeros of $q-\psi(z)$ are real and satisfy the interlacing property. Therefore, we may re-arrange (4.19) in such a way that we recognize the Mellin transform of a product of gamma and beta random variables. The uniqueness of the Mellin transform then gives us the result of Theorem 14. However, when $\hat{\rho}_2 \notin \mathbb{R}$, we are not assured of real solutions and poles, nor of the interlacing property. Therefore, inverting the Mellin transform in the manner that we have just described might lead to a product of gamma and beta random variables with negative or complex parameters (cf. (4.6)). Since such an object is not defined, we have to resort to other methods.

Suppose now that $\hat{\rho}_2 \notin \mathbb{R}$, so that the process X is no longer a hyper-exponential process. We may find the density of the distribution of $I_q(X)$ by inverting the Mellin transform using the Bromwich integral

$$\begin{aligned} p(x) &= \frac{1}{2\pi i}\int_{1+i\mathbb{R}} h(z)x^{-z}\mathrm{d}z \\ &= \frac{\sigma^2}{2\mathcal{G}(1)}\int_{1+\imath\mathbb{R}} \frac{\prod_{j=1}^2\Gamma(\zeta_j-z+1)\times\prod_{j=1}^2\Gamma(z+\hat{\rho}_j)\times\Gamma(z)}{\prod_{j=1}^3\Gamma(z+\hat{\zeta}_j)\times\Gamma(\rho_1-z+1)}\left(\left(\frac{\sigma^2}{2}x\right)^{-1}\right)^z \mathrm{d}z. \end{aligned} \tag{4.21}$$

To arrive at the result of Theorem 15 we need simply to compare the integral expression in (4.21) with the definition of the Meijer-G function in Appendix A. It is easy to see that the integral is equal to $\mathbf{G}^{2,3}_{4,5}\left((\frac{\sigma^2}{2}x)^{-1}\middle|\begin{smallmatrix}1,1-\hat{\rho}_1,1-\hat{\rho}_2,1+\rho_2\\1+\zeta_1,1+\zeta_2,1-\hat{\zeta}_1,1-\hat{\zeta}_2,1-\hat{\zeta}_3\end{smallmatrix}\right)$. The technical conditions in the definition of the Meijer-G function regarding the poles of the integrand are satisfied due to our assumption (see the assumption prior to Theorem 15) that the function $\Gamma(z)\mathcal{G}(z)$ has only simple poles.

4.2.4 The Mellin transform and Asian options

In Section 4.3.3 we consider the case where X is a scaled Brownian motion with drift. We find that we may derive a semi-explicit expression for the function $h_a(k,q) = \mathbb{E}[(I_q(X)-k)^+]$ (see 4.3). The reason we are able to do this, is that the density of the distribution of $I_q(X)$ is

tractable enough to obtain the result directly. When this is not the case, i.e. if the density is not known or is too complicated, we may resort to another method involving the Mellin transform which was pioneered by Cai and Kou [27]. Our goal is to show that the Mellin transform in the k variable of the function $h_a(k,q)$, which is defined as $\Phi(z,q) := \int_{\mathbb{R}^+} h_a(k,q)k^{z-1}\mathrm{d}k$, can be expressed in terms of $\mathcal{M}(I_q, z)$. Indeed, provided $\mathcal{M}(I_q, z+2)$ is finite on some strip $0 < \mathrm{Re}(z) < \alpha$, then for $\mathrm{Re}(z) \in (0,\alpha)$ we have,

$$\begin{aligned}\Phi(z,q) = \mathbb{E}\left[\int_{\mathbb{R}^+} (I_q - k)^+ k^{z-1}\mathrm{d}k\right] &= \mathbb{E}\left[\int_0^{I_q} (I_q - k)\, k^{z-1}\mathrm{d}k\right] \\ &= \frac{\mathbb{E}\left[I_q^{z+1}\right]}{z(z+1)} = \frac{\mathcal{M}(I_q, z+2)}{z(z+1)}.\end{aligned} \tag{4.22}$$

Therefore, the algorithm to recover the price, or equivalently the function $f_a(k,t) = \mathbb{E}[(I_t - k)^+]$ proceeds along the following lines: a) Derive an expression for $\Phi(z,q)$; and b) apply a Mellin-Laplace inversion to recover $f_a(k,t)$. We see from (4.22) that the nature of this procedure depends on the form of $\mathcal{M}(I_q, z)$. Likely, we will not be able to find the inverse analytically, so the question becomes: How amenable is the expression of $\mathcal{M}(I_q, z)$ to numerical inversion? We will discuss the technical aspects of numerical inversion in Chapter 5.

4.3 Connection with pssMps and the Lamperti transform

In 1972 Lamperti [80] discovered an important connection between Lévy processes and the class of positive self-similar Markov processes which relies on, and simultaneously gives insight into, the exponential functional. We summarize this finding in Section 4.3.2, using the approach of [77] and [42], and present three interesting examples of its application in the literature in Section 4.3.3. In order to properly present these ideas we will need some theory and notation from the field of Markov processes, which we present first.

4.3.1 Markov processes: A brief review

A ***Markov process*** is a stochastic process X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that,

$$\mathbb{P}(X_{s+t} \in B | \mathcal{F}_t) = \mathbb{P}(X_{s+t} \in B | X_t) \tag{4.23}$$

holds for all $0 \leq s$, t and Borel subsets B of the state space S. We will assume that is S is equal to $\mathbb{R}^+$, $\bar{\mathbb{R}}^+$, or $\mathbb{R}$. Now suppose τ is a stopping time for the filtration $\mathbb{F}$, and define the σ-algebra $\mathcal{F}_\tau := \{\mathcal{A} \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$. If, on the event $\{\tau < \infty\}$, X satisfies (4.23) for $t = \tau$, then X is called a ***strong Markov processes***.

It is straightforward to verify that Lévy processes are Markov processes from the fact that Lévy processes have independent increments (property (iii) in the definition in Section 2.1.3). From [76] we have the following much stronger result, which implies that Lévy processes are, in fact, strong Markov processes.

Theorem 17 (Theorem 3.1 in [76]). *Suppose that τ is a stopping time. Define on $\{\tau < \infty\}$ the process $\tilde{X}$, where*

$$\tilde{X}_t := X_{\tau+t} - X_\tau, \quad t \geq 0.$$

Then on the event $\{\tau < \infty\}$ the process $\tilde{X}$ is independent of $\mathcal{F}_\tau$, and has the same law as X. In particular $\tilde{X}$ is a Lévy process.

In what follows, we will work with collections of Markov processes with state space $\bar{\mathbb{R}}^+$. These will be indexed by $x \in \mathbb{R}^+$, where the index x also denotes the initial position of the process. To refer to such a collection we will write $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$, where for each x: a) $X^{(x)}$ is a Markov process under $\mathbb{P}_x$; and b) $\mathbb{P}_x(X_0^{(x)} = x) = 1$.

We recall that a killed Lévy process is a Lévy process that is sent to a cemetery state Δ at an independent exponential time $\mathbf{e}(q)$. If we allow for the convention the $\mathbf{e}(0) = \infty$ then the collection of killed Lévy processes includes all Lévy processes. When we refer to a killed Lévy process in Section 4.3, we allow for the possibility that $q = 0$, and we set $\Delta = -\infty$. As in Section 2.1.3 we denote by ζ the lifetime of the process, i.e. the time until it reaches $-\infty$. We can extend this idea to the Markov processes $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$. We say such a process $X^{(x)}$ is ***killed*** if we allow for the possibility that the process leaves the state space and reaches zero, remaining there $\mathbb{P}_x$-a.s afterwards. In this case, we call the point zero a ***cemetery state*** for the process $X^{(x)}$. We say the collection $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ is ***killed***, if for each x, zero is an cemetery state for $X^{(x)}$. The reader should note that the ***lifetime*** of the process $X^{(x)}$, which we define by

$$T_0^x := \inf\{t > 0 : X_t^{(x)} = 0\}$$

is not necessarily an exponential random variable. However, just like in the case of a Lévy

process, we allow for the possibility that a killed process has the property: $T_0^x = \infty$ almost surely.

Now we are ready to introduce the principal object of this section, namely, the positive self-similar Markov process. A ***positive self-similar Markov process (pssMp)*** is a collection of processes $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ on state space $\bar{\mathbb{R}}^+$ such that: $\mathbb{P}_x(X_0^{(x)} = x) = 1$; $(X^{(x)}, \mathbb{P}_x)$ is a strong Markov process; and for some $\alpha > 0$ and every $k > 0$ the ***self-similarity property***, i.e.

$$\text{the distribution of } \{kX^{(x)}_{k^{-\alpha}t} : t \geq 0\} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx},$$

holds. The constant α is known as the ***index of self-similarity***.

4.3.2 The Lamperti transform

The Lamperti transform defines a bijection between the class of killed Lévy processes and the class of killed pssMps. Although we will not give a proof for the general case, we will demonstrate the underlying ideas using the example of a Brownian motion with drift. The content of this example is borrowed from [42]. We will show that if we exponentiate our process, and apply the correct time change, then we obtain a Bessel process (a pssMp). It turns out that this method of "exponentiation plus time change" works in the general case as well and is known as the Lamperti transform.

First we give the necessary details on Bessel processes. These facts are taken from Chapter XI in [99]. A ***Bessel process*** with dimension $\delta \geq 2$ is a solution of the equation

$$X_t^{(x)} = x + \frac{\delta - 1}{2}\int_0^t (X_s^{(x)})^{-1}\mathrm{d}s + B_t, \tag{4.24}$$

where $x > 0$ and $B = \{B_t : t \geq 0\}$ is a Brownian motion. Such a process is a strong Markov process, started at x, with transition density

$$p_t^{(\nu)}(w, z) := \frac{z}{t}\left(\frac{z}{w}\right)^{\nu} e^{-(w^2+z^2)/2t}\mathcal{I}_\nu(\tfrac{wz}{t}), \quad t,\, w,\, z > 0,$$

where $\nu = \delta/2 - 1$, and $\mathcal{I}_\nu(z)$ is the modified Bessel function of the second kind. By transition density, we mean that $p_t^{(\nu)}(w, z)$ is a function that satisfies

$$\mathbb{P}_x(X_t^{(x)} \in A) = \int_A p_t^{(\nu)}(x, z)\mathrm{d}z \tag{4.25}$$

for any $t \geq 0$ and $A \in \mathcal{B}_{\mathbb{R}}$. One may verify that $p_t^{(\nu)}(kw, z) = k^{-1} p_{k^{-2}t}^{(\nu)}(w, k^{-1}z)$, which shows, together with (4.25), that a collection of Bessel processes, $\{(X^{(x)}, \mathbb{P}_x)\}_{x \in \mathbb{R}^+}$, with common dimension $\delta \geq 2$ is a pssMp with index of self-similarity $\alpha = 2$.

We may also define Bessel processes for $0 \leq \delta < 2$, but in those cases we must use a more general definition which does not suit our present purpose. We mention this only to point out that the dimension δ determines the time it takes the process to visit zero for the first time. That is, when $0 \leq \delta < 2$, then $T_0^x < \infty$, $\mathbb{P}_x$-a.s.; however, when $\delta \geq 2$ we have $T_0^x = \infty$, $\mathbb{P}_x$-a.s..

Now, let us return to our example. Let ξ be a Brownian Motion with drift $\nu \geq 0$, i.e. $\xi_t := \nu t + B_t$ where B is a standard Brownian motion. Our goal is to transform ξ into a Bessel process $X^{(x)}$. Since we know that such a process is positive with initial position $x > 0$, a natural idea is to exponentiate ξ and then scale by x. After an application of Itô's Formula, this procedure gives

$$xe^{\xi_t} = x + \left(\nu + \frac{1}{2}\right) x \int_0^t e^{\xi_s} \mathrm{d}s + M_t, \qquad \text{where} \qquad M_t := x \int_0^t e^{\xi_s} \mathrm{d}B_s. \tag{4.26}$$

Comparing (4.24) and (4.26) we see that our transformation did not entirely accomplish our goal. To complete the transformation to a Bessel process, we need to implement a time change. We notice that the exponential functional $I_t(\xi)$ appears in (4.26); our time change will involve the inverse of $I_t(2\xi)$ (note that $\alpha = 2$ for Bessel processes). That is, we will define the function $\tau(u)$ by

$$\tau(u) := \inf\{s \geq 0 : I_s(2\xi) > u\}.$$

Since $\nu \geq 0$, and therefore $\lim_{t \to \infty} I_t(2\xi) = +\infty$, the function $\tau(u)$ is finite for all $u \geq 0$, which means that $I_{\tau(u)}(2\xi) = u$ for all $u \geq 0$. Differentiating this last equality gives

$$\frac{\mathrm{d}}{\mathrm{d}u}\tau(u) = e^{-2\xi_{\tau(u)}}. \tag{4.27}$$

We claim that the process

$$X_t^{(x)} := x \exp(\xi_{\tau(x^{-2}t)}) \tag{4.28}$$

is a Bessel process with dimension $\delta = 2(\nu + 1)$. To verify this claim we substitute $\tau(x^{-2}t)$ for t in (4.26) and check how our time change affects the last two terms on the right-hand

side. For the first term, a change of variables, $s \mapsto \tau(x^{-2}s)$, and an application of (4.27) shows that

$$x\int_0^{\tau(x^{-2}t)} e^{\xi_s}\mathrm{d}s = \int_0^t \frac{\mathrm{d}s}{x\exp(\xi_{\tau(x^{-2}s)})} = \int_0^t (X_s^{(x)})^{-1}\mathrm{d}s. \tag{4.29}$$

For the second term, we note that M_t is a continuous martingale with quadratic variation $[M,M]_t = x^2 I_t(2\xi)$ [3]. A well-known theorem (see the Dambis, Dubins-Schwartz Theorem in [99]) tells us that there exists a Brownian motion $\tilde{B}$ such that $M_t = \tilde{B}_{[M,M]_t}$. This shows that $M_{\tau(x^{-2}t)} = \tilde{B}_t$, which, together with (4.29), proves that $X^{(x)}$ satisfies equation (4.24). Therefore, $X^{(x)}$ is a Bessel process with initial position x.

We may also reverse the transformation. Integrating (4.27) gives

$$\tau(t) = \int_0^{x^2 t} (X_s^{(x)})^{-2}\mathrm{d}s,$$

so that we recover ξ by the formula

$$\xi_t = \log\left(\frac{X^{(x)}_{x^2\tau^{-1}(t)}}{x}\right). \tag{4.30}$$

We have presented arguably the most basic example of the Lamperti transform. Our analysis was simplified because we chose to work with continuous processes. The situation becomes more complicated when we involve Lévy processes and pssMps with discontinuous paths. Besides path continuity, we incorporated another simplifying assumption which bears discussion. Specifically, we chose ξ such that $\zeta = \infty$, and $\limsup_{t\to\infty}\xi_t = +\infty$, which had the effect that $I_{\tau(u)}(2\xi) = u$ for all $u \geq 0$. The example can easily be generalized for the scenarios: a) $\zeta < \infty$; and b) $\zeta = \infty$ and $\lim_{t\to\infty}\xi_t = -\infty$. However, in these cases we have $I_\infty(2\xi) < \infty$ so that $\tau(t) = \infty$ for $t \geq I_\infty(2\xi)$. Applying transformation (4.28) yields a killed pssMp which reaches zero in the finite time $x^2 I_\infty(2\xi)$ and remains at zero thereafter. The manner in which the pssMp reaches zero, either by jumping, or in a continuous fashion, depends on the Lévy process: scenario a) leads to the former behaviour, while b) leads to the latter. Likewise, we could have started with a Bessel process with dimension $\delta < 2$. Then the process defined by $X_t^{(x)}\mathbb{I}(t \leq T_0^x)$ would have, via the inverse transformation (4.30), yielded a Lévy process for which either: $\zeta < \infty$, or $\zeta = \infty$ and $\lim_{t\to\infty}\xi_t = -\infty$.

[3] The reader unfamiliar with the concept of martingales and quadratic variation is referred to [99]

The important conclusion, however, is that essentially the same transformation applies in the general setting. Suppose that $\alpha > 0$ and ξ is a killed Lévy process, and define $\tau(t)$ to be

$$\tau(t) := \inf\{s \geq 0 : I_s(\alpha\xi) > t\}. \tag{4.31}$$

We define the ***Lamperti transform*** of ξ to be the process $X^{(x)}$ given by

$$X_t^{(x)} := x \exp(\xi_{\tau(x^{-\alpha}t)}). \tag{4.32}$$

If we apply the transform for all $x \in \mathbb{R}^+$, we get a collection $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ which is a killed pssMp with index of self-similarity equal to α. Conversely, starting with a killed pssMp $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ we may apply the inverse transform

$$\xi_t = \log\left(\frac{X^{(x)}_{x^\alpha\gamma(t)}}{x}\right), \quad \text{where} \quad \gamma(t) = \inf\left\{s \geq 0 : \int_0^{x^\alpha s} (X_s^{(x)})^{-\alpha}\mathrm{d}s > t\right\},$$

to any of the processes $X^{(x)}$, to obtain a killed Lévy process ξ. The process ξ will be the same for each x.

The formal statement of this result follows. It also includes a classification of the processes ξ and $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ according to their lifetimes, and makes an important connection between the random variables T_0^x and $I_q(\alpha\xi)$. We reproduce this theorem here from [77]; the result is originally due to Lamperti [80].

Theorem 18 (Theorem 13.1 in [77])**.** *Fix $\alpha > 0$.*

(i) If $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ is a killed pssMp with index of self-similarity α, then there exists a killed Lévy process ξ such that $X^{(x)}$ can be represented by the Lamperti transform of ξ. Either:

(a) $T_0^x = \infty$, $\mathbb{P}_x$-a.s. for all $x > 0$, in which case $\zeta = \infty$ and ξ satisfies $\limsup_{t\to\infty} \xi_t = \infty$;

(b) $T_0^x < \infty$ and $X_{T_0^x-} = 0$, $\mathbb{P}_x$-a.s. for all $x > 0$, in which case $\zeta = \infty$ and ξ satisfies $\lim_{t\to\infty} \xi_t = -\infty$; or

(c) $T_0^x < \infty$ and $X_{T_0^x-} > 0$, $\mathbb{P}_x$-a.s. for all $x > 0$, in which case ξ is a Lévy process killed at rate $q > 0$.

For all cases $T_0^x \stackrel{d}{=} x^\alpha I_q(\alpha\xi)$.

(ii) Conversely, suppose that ξ is a killed Lévy process. Then the family, $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}^+}$ defined by the Lamperti transform is a killed pssMp with index of self-similarity α. The random variable T_0^x satisfies $T_0^x \stackrel{d}{=} x^\alpha I_q(\alpha\xi)$.

4.3.3 Applications of the Lamperti transform

Example 1: Asian options under the Black-Scholes model

In this section we present a slight variation of our example of Section 4.3.2. We now assume that ξ is a be a *scaled* Brownian motion with drift, i.e.

$$\xi_t := \mu t + \sigma B_t, \quad t \geq 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and B is a standard Brownian motion. We present two related results: the first, gives the distribution of the exponential functional $I_q(\xi)$; the second, gives a semi-explicit formula for the price of an Asian option for a stock price driven by ξ. A nice summary of this theory, with detailed proofs, is given in Dufresne [42]. We follow the presentation of [42] below, but the reader should note the main results are due to other authors; we will cite these as necessary.

Before stating any results we demonstrate that it is sufficient to consider the case $\sigma = 1$. We may do this, without loss of generality, in the following fashion: for $\nu \in \mathbb{R}$ define the random variable

$$J_u^\nu := \int_0^u e^{2(\nu t + B_t)} \mathrm{d}t.$$

Now we observe that a change of variables $t \mapsto 4s/\sigma^2$, and the self-similarity property of Brownian motion, imply that

$$I_u(\xi) = \int_0^u e^{\mu t + \sigma B_t} \mathrm{d}t \stackrel{d}{=} \frac{4}{\sigma^2} J_\gamma^\eta, \tag{4.33}$$

where $\gamma = \sigma^2 u/4$, and $\eta = 2\mu/\sigma^2$. Thus, we may concentrate on the random variable J_u^ν which we observe is just the exponential functional (with $\alpha = 2$ and non-random time u) of a standard Brownian motion with drift v. The key result, due to Yor, is that we may determine the distribution of $J_{\mathbf{e}(q)}^\nu$ in terms of the distributions of well known random variables.

Theorem 19 (Theorem 2 in [113]). *Suppose $J^{\nu}_{e(q)}$ is as defined above, and set*

$$\alpha = \frac{\gamma + \nu}{2}, \quad \beta = \frac{\gamma - \nu}{2}, \quad \gamma = \sqrt{2q + \nu^2}.$$

Then

$$J^{\nu}_{e(q)} \stackrel{d}{=} \frac{B_{(1,\alpha)}}{2G_{(\beta,1)}}$$

where $B_{(1,\alpha)}$ is a beta random variable with parameters 1, and α, and $G_{(\beta,1)}$ is a gamma random variable with parameters 1 and β, and $B_{(1,\alpha)}$ and $G_{(\beta,1)}$ are independent.

The proof of Theorem 19 relies critically on the Lamperti transform. Specifically, it relies on the fact that for $\nu \geq 0$, $J^{\nu}_{\tau(t)}$ is a Bessel process with initial position one and dimension $2(\nu + 1)$; we showed this in Section 4.3.2. Another possibility for obtaining the result is to view $2(\nu t + B_t)$ as a hyper-exponential process without jumps and then use Theorem 14.

Theorem 19 also yields a method for pricing Asian options. Using (4.33) and the following theorem, which we may prove by direct calculation from Theorem 19, we may easily recover the function $h_a(k, q)$ and subsequently the option price. We assume now that ξ is defined with respect to a risk neutral measure $\mathbb{P}$.

Theorem 20 (Formula 3.10 in [114]). *For all $\nu \in \mathbb{R}$, $k > 0$, and $q > 2(\nu + 1)$,*

$$\int_{\mathbb{R}^+} e^{-qt}\mathbb{E}[(J^{(\nu)}_t - k)^+]\mathrm{d}t = \frac{(2k)^{1-\beta}}{2q(\alpha+1)\Gamma(\beta)} \int_0^1 u^{\beta-2}(1-u)^{\alpha+1}e^{-u/2k}\mathrm{d}u, \tag{4.34}$$

where α and β are defined as in Theorem 19.

Finally, we give an alternative formulation of (4.34) in terms of the confluent hypergeometric function. This was developed by Donati-Martin et. al. [37]:

$$\int_{\mathbb{R}^+} e^{-qt}\mathbb{E}[(J^{(\nu)}_t - k)^+]\mathrm{d}t = \frac{(2k)^{1-\beta}\Gamma(\alpha+1)}{2q(\beta-1)\Gamma(\alpha+\beta+1)}\ {}_1F_1\left(\beta - 1; \alpha + \beta + 1; \frac{1}{2k}\right).$$

Example 2: Density of the supremum of a stable process

In this example, which we take from [72], we show how to derive a formula for the density of the supremum of a stable process via the Lamperti transform of a hypergeometric Lévy process. We will see in Chapter 7 that we may also approach this problem via the Wiener-Hopf factorization for stable processes. The technique we present below is much simpler

than the one we present in Chapter 7. The caveat is that we must know the distribution of the exponential functional of a hypergeometric process in order to apply it–and, deriving the distribution is not a trivial task. We include this example because it lies in the intersection of the results we may derive via Wiener-Hopf techniques, and the results we may derive via techniques based on the exponential functional.

Before giving our example, we take a brief detour to introduce hypergeometric Lévy processes. A ***hypergeometric process*** is a Lévy process with Laplace exponent

$$\psi(z) = -\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)},$$

where $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ belongs to the admissible set of parameters

$$\mathcal{D} = \{\beta \leq 1,\, \gamma \in (0,1),\, \hat{\beta} \geq 0,\, \hat{\gamma} \in (0,1)\}.$$

The reader is referred to [72] for a proof of the fact that a process so defined is, in fact, a Lévy process. Once this has been established, we readily see from the Laplace exponent that hypergeometric processes are meromorphic processes. This is not surprising, as they are constructed according to Vigon's theory of philanthropy [111] which entails gluing together the Laplace exponents of two β-subordinators. The resulting function is the Laplace exponent of a hypergeometric process, whose ascending and descending ladder height processes are again β-subordinators. A direct consequence of this construction is that we may express the Wiener-Hopf factors of hypergeometric processes in closed form.

We return now to our problem, and assume that Y is a stable process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with admissible parameters (α, ρ) (see (2.8)). We wish to study the density $p(x)$ of the distribution of the running supremum process S at time 1. As usual, we define S by the equality $S_t = \sup\{Y_s : 0 \leq s \leq t\}$. To proceed, let $X_t^{(x)} = (x - Y_t)\mathbb{I}(t < \tilde{T}_0^x)$, where $\tilde{T}_0^x = \inf\{t \geq 0 : x - Y_t \leq 0\}$. Then, the self-similarity property of stable processes (2.6), and the fact that Lévy processes are strong Markov processes, together imply that $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}_+}$ is a pssMp, and $\tilde{T}_0^x \overset{d}{=} T_0^x$.

To reach our desired result, we rely on the following facts:

(a) From Proposition 1, and Theorem 1 in [72], we know that the Lévy process ξ corresponding to the pssMp $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}_+}$ via the Lamperti transform is a killed hypergeometric

process with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1 - \alpha\rho, \alpha(1-\rho), 1 - \alpha\rho, \alpha\rho),$$

and killing rate $q = \Gamma(\alpha)/(\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)) \geq 0$ [4]. For the remainder of this example q will refer to this specific value, and $\mathbf{e}(q)$ will be the exponential random variable with this parameter q. The reader should note that this process is a member of the Lamperti-stable group of processes first introduced in [23].

(b) Let Ξ be identical to ξ except with lifetime $\zeta = \infty$; that is, if ξ has Laplace exponent $\psi(z)$, then Ξ is the process with Laplace exponent $\psi(z) + q$. According to Theorem 4 in [72], for $\alpha \notin \mathbb{Q} \cup \mathcal{L}$ we have an absolutely convergent series expression for the density $p(x)$ of $I_q(\alpha\Xi)$. Here $\mathcal{L}$ is the set of irrational numbers defined by (3.11).

(c) Theorem 18 tells us that $\tilde{T}_0^1 \stackrel{d}{=} I_q(\alpha\Xi)$.

Now, we just need to arrange these facts in the correct way and use the self-similarity property of stable processes. That is,

$$\mathbb{P}(\tilde{T}_0^1 < t) = \mathbb{P}(\sup\{Y_s : 0 \leq s \leq t\} > 1) = \mathbb{P}(S_t > 1) = \mathbb{P}(t^{1/\alpha} S_1 > 1) = \mathbb{P}(S_1 > t^{-1/\alpha}).$$

The above calculation shows that we may derive the distribution of S_1^{-a} in terms of the distribution of $I_q(\alpha\xi)$. Straightforward computation then allows us to find the density of S_1 in terms of the density $p(x)$. Specifically,

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(S_1 < x) = \alpha x^{-1-\alpha} p(x^{-\alpha}), \quad x > 0.$$

Example 3: The homogeneous functional $\mathcal{A}_r$

Consider a stable process Y with admissible parameters (α, ρ), and denote by $L^{(x)}$ the process

$$L_t^{(x)} := x + Y_t, \quad x, t > 0.$$

As in the previous example, we can construct a pssMp by defining $X_t^{(x)} := L_t^{(x)} \mathbb{I}(t \leq \tilde{T}_0^x)$, where $\tilde{T}_0^x := \inf\{t > 0 : L_t^{(x)} < 0\}$. Again we have $T_0^x \stackrel{d}{=} \tilde{T}_0^x$.

We are interested in the case $x = 1$, and specifically we are interested in the following

[4] Observe that $1 - \alpha\rho$ could potentially be 0, so we may face the situation $q = 0$. In this case Proposition 1 in [72] assures us that ξ drifts to $-\infty$, so that the exponential functional $I_q(\alpha\xi)|_{q=0} := I_\infty(\alpha\xi)$ is well-defined.

homogeneous functional

$$\mathcal{A}_r := \int_0^{\tilde{T}_0^1} |L_t^{(1)}|^r \mathrm{d}t, \quad r \in \bar{\mathbb{R}}^+.$$

The random variable $\mathcal{A}_r$ was recently studied in [81] and is clearly an important and versatile object. When $r = 0$ it is equal to the first passage time $\tilde{T}_0^1$; when $r = 1$ it is equal to the area below the graph of $L_t^{(1)}$, $0 \leq s \leq \tilde{T}_0^1$; finally, when $r \to +\infty$ the random variable $\mathcal{A}_r^{1/r}$ converges in distribution to the supremum of $L^{(1)}$ before time $\tilde{T}_0^1$.

Our goal is to demonstrate that we can easily determine the distribution of $\mathcal{A}_r$ in terms of the distribution of the exponential functional of the hypergeometric process associated with $\{(X^{(x)}, \mathbb{P}_x)\}_{x\in\mathbb{R}+}$ via the Lamperti transform. The distribution of $\mathcal{A}_r$ was originally identified in [81]; our approach via the exponential functional is different than the one presented there. The key to our method is to use Theorem 1 in [72] once more. This states that the associated Lévy process ξ is a killed hypergeometric process with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1 - \alpha(1-\rho), \alpha\rho, 1 - \alpha(1-\rho), \alpha(1-\rho)),$$

and killing rate $q = \Gamma(\alpha)/(\Gamma(\alpha(1-\rho))\Gamma(1-\alpha(1-\rho)) \geq 0$. As before, let Ξ be identical to ξ except with lifetime $\zeta = \infty$. Then we have

$$\mathcal{A}_r \stackrel{d}{=} \int_0^{T_0^1} |L_t^{(1)}|^r \mathrm{d}t \stackrel{d}{=} \int_0^{I_q(\alpha\xi)} e^{r\xi_{\tau(t)}} \mathrm{d}t = \int_0^{\mathbf{e}(q)} e^{(\alpha+r)\Xi_s} \mathrm{d}s,$$

where the second equality in distribution follows from Theorem 18 and the last equality follows from a change of variables $t \mapsto I_s(\alpha\xi)$ and an argument similar to that used in the derivation of (4.27). In summary, $\mathcal{A}_r$ is equal in distribution to the exponential functional $I_q((\alpha + r)\Xi)$. In Chapter 6 we derive an explicit expression I_q as an infinite product of independent beta random variables. As a simple corollary (Corollary 1) we can show that $\mathcal{A}_r$ is also distributed like such an infinite product.

Chapter 5

Numerical Techniques

We discuss two numerical techniques for evaluating a particular type of oscillatory integral that arises naturally in the option pricing problems we consider in Chapters 6 and 8. The first is a transform method known informally as "changing the contour of integration", and the second is method of numerical integration known as Filon's method. Both of these techniques are demonstrated via examples. The option pricing problems of Chapters 6 and 8 will also require that we find solutions of the equation $\psi(z) = q$. In the context of meromorphic and hyper-exponential processes we consider the cases: a) where the solutions become large; and b) where $q \in \mathbb{C}$. We discuss potential problems that arise in these two cases and also their solutions. In the second part of the chapter we present an important connection between Pick functions, Stieltjes functions, and the Laplace exponent of a Lévy process with completely monotone jumps. Additionally, we give some remarkable facts about rational approximations of Pick and Stieltjes functions with a special focus on Padé approximants. This theory is important for the results we develop in Chapter 8.

5.1 Introduction

There are two objectives for this chapter. The first, is to demonstrate some techniques for numerical evaluation of integrals of the type

$$\int_a^b f(u)e^{iux}\mathrm{d}u, \quad -\infty \leq a \leq b \leq \infty. \tag{5.1}$$

We are faced with the task of computing such integrals both in Chapter 6 and Chapter 8 when we calculate option prices and cumulative distribution functions. There are two

steps to computing such integrals: 1) perform simplifying transformations to ensure that the integral (5.1) converges as quickly as possible; and 2) choose a numerical method which deals with the oscillatory nature of the integral effectively. In Section 5.2 we will discuss one transformation method, and one numerical method (both methods are relatively well-known to numerical analysts) as this is sufficient for our purposes. In Section 5.2.3 we then consider the special case where $f(z)$ is parameterized by the solutions of the equation $\psi(z) = q$. In particular, we assume that $\psi(z)$ is the Laplace exponent of either a hyper-exponential or meromorphic process and discuss potential difficulties in finding large solutions, or solutions for the case $q \in \mathbb{C}$. We also explain why we will encounter the latter scenario (recall that q is usually real).

The second objective for this chapter is to introduce some ideas from the theory of rational approximation. In Chapter 8 we approximate the Laplace exponents of completely monotone processes by rational functions. Our technique ensures that the resulting rational functions are the Laplace exponents of hyper-exponential processes. The basis for this procedure is rooted in the theory of Padé approximants of Stieltjes functions. We present the key aspects of this theory in Section 5.3.

5.2 Techniques for numerical integration

5.2.1 Transforming the integral: Changing the contour of integration

In the numerical setting we can only deal with finite quantities, in particular, we may need to truncate the domain of integration of (5.1). In doing so we can encounter the problem that our integrand $f(u)e^{iux}$ decays too slowly. In such cases, unless we can transform (5.1), we are forced to use a large domain of numerical integration which leads to increased computational effort, and potentially error.

In this section, we demonstrate through an example, a technique that is effective in modifying an integral like (5.1) so that the resulting integrand decays more rapidly. This technique, which we call "changing the contour of integration", is essentially just an application of Cauchy's integral theorem. The method is accurately described by its name: we replace the contour of integration by another, and show that the resulting expressions are equal. The integrand of the new integral decays more rapidly than the integrand of the first; this means that we may restrict our numerical scheme to a smaller interval.

We now turn to our example. Consider the Gamma subordinator X with Laplace exponent

$$\psi(z) = -\frac{1}{2}\log\left(1 - \frac{z}{\rho}\right), \quad 0 < \rho,\ \mathrm{Re}(z) < \rho.$$

Suppose we want to calculate the CDF of X_1, which we call $F(x)$, by numerical means. Fairly straightforward arguments show that for $0 < c < \rho$ we have $F(x) = 1 - G(x)$ where

$$G(x) = \frac{e^{-cx}}{2\pi} I(x), \quad I(x) = \int_{\mathbb{R}} f(c+iu)e^{-iux}\mathrm{d}u, \quad \text{and} \quad f(z) = \frac{e^{\psi(z)}}{z}. \tag{5.2}$$

We see that the integrand $f(z)$ decays as follows:

$$|f(c+iu)e^{-iux}| \sim \sqrt{\rho}|u|^{-3/2}, \quad |u| \to \infty. \tag{5.3}$$

We can argue quite easily that such a rate of decay is too slow. For example, suppose we choose a naïve numerical scheme (we will see in the next section that this scheme is inadequate, we use it here just as an example) to evaluate $I(x)$. We define the ***right endpoint scheme*** to be the scheme which approximates $I(x)$ as follows:

$$I(x) \approx \sum_{n=1}^{2N} \Delta f(c + i(n\Delta - L))e^{-i(n\Delta - L)x}.$$

The scheme is implemented on the domain of numerical integration $[-L, L]$ which is divided into $2N$ intervals of equal size Δ. If we set $L = 100$ then we might expect from (5.3) that $|f(c+i100)e^{-i100x}|$ is of the order 10^{-3}. This is a significant quantity if we would like our calculation to be accurate to two or three decimal places. Therefore, unless we can transform the integral somehow, we have to expand the domain of numerical integration, and, assuming we keep Δ constant, perform more calculations.

To demonstrate the technique of changing the contour of integration let us first re-write $G(x)$ in a more convenient form. We make a change of variables $u \mapsto -i(z-c)$ in $I(x)$ and use the fact that $\overline{f(z)} = f(\bar{z})$ in order to write

$$G(x) = \frac{1}{\pi}\mathrm{Re}\left(\int_{c+i\mathbb{R}^+} f(z)e^{-zx}\mathrm{d}z\right). \tag{5.4}$$

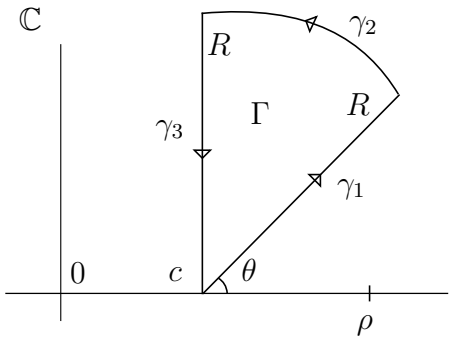


Figure 5.1: We integrate over the curve Γ for the first integral in (5.4). When $R \to \infty$ the integral over γ_2 vanishes.

Now, we consider the new integral in (5.4). Instead of integrating over the line $c + i\mathbb{R}^+$, suppose we integrate over the closed contour Γ pictured in Figure 5.1. Cauchy's integral theorem tells us that

$$\int_{\gamma_1} f(z)e^{-zx}\mathrm{d}z = -\int_{\gamma_2} f(z)e^{-zx}\mathrm{d}z - \int_{\gamma_3} f(z)e^{-zx}\mathrm{d}z.$$

It is not difficult to see that the integral over the contour γ_2 vanishes as $R \to \infty$. This shows that

$$\int_{c+e^{i\theta}\mathbb{R}^+} f(z)e^{-zx}\mathrm{d}z = \int_{c+i\mathbb{R}^+} f(z)e^{-zx}\mathrm{d}z,$$

in other words, we may change the contour of integration from $c+i\mathbb{R}^+$ to $c+e^{i\theta}\mathbb{R}^+$. Making one more change of variables $z \mapsto c + e^{i\theta}u$ gives

$$G(z) = \frac{e^{-cx}}{\pi}\mathrm{Re}\left(e^{i\theta}\int_{\mathbb{R}^+} f(c+e^{i\theta}u)e^{-i\sin(\theta)ux}e^{-\cos(\theta)ux}\mathrm{d}u\right). \tag{5.5}$$

We may verify that as $|u|$ grows the modulus of the integrand in (5.5) behaves like $|u|^{-3/2}$ $\times\, e^{-\cos(\theta)|u|x}$ instead of like $|u|^{-3/2}$. This makes quite a difference in our numerical scheme. For example, suppose $x = 1$, and $\theta = \pi/4$. Then we would expect $|f(c + e^{i\theta}100)e^{-i\sin(\theta)ux}$ $\times\, e^{-\cos(\theta)100x}|$ to be of the order 10^{-33}, or virtually zero. That is, a domain of numerical integration of $[-100, 100]$ is more than large enough to handle the transformed problem.

For this particular problem it is easy to see that θ should take values in $(0, \pi/2)$, and, of course, the closer that θ is to zero, the faster the rate of decay of the integrand. However, we also note that as θ approaches zero, our integrand becomes unbounded near the singu-

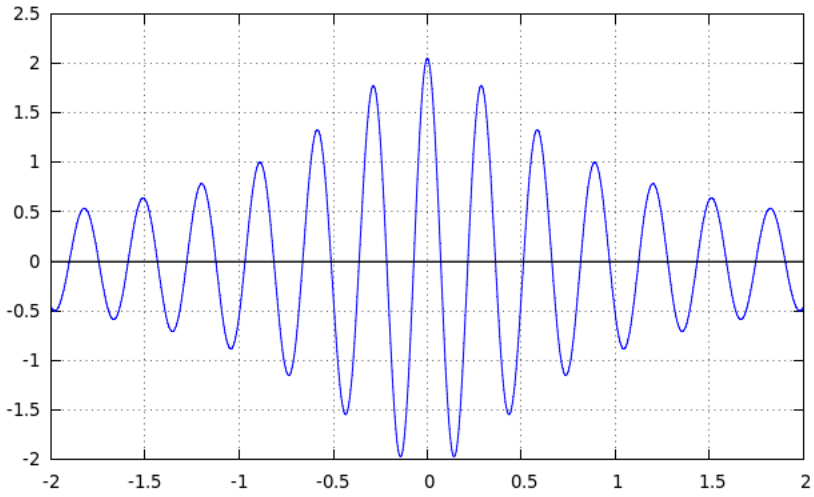


Figure 5.2: Real part of the integrand of $I(x)$ near $u = 0$. In this scenario $x = 20$ and the parameters are $\rho = 10$, and $c = 1/2$.

larity $z = \rho$. Our choice of θ should therefore strike a balance between maximizing the rate of decay of the integrand, and avoiding the singularity at ρ.

5.2.2 Choosing a numerical method: Filon's method

In this section we show that neither the right endpoint scheme, nor its generalization, the Newton-Cotes scheme, is actually an appropriate method to approximate the integral $I(x)$ from (5.2). The reason such standard approaches to numerical integration fail is that they deal poorly with the oscillatory nature of the integrand, and, in particular, with the fact that the frequency of the oscillations is x-dependent. At the end of the section we discuss a well-known method – Filon's method [44,45] – that is accurate for oscillatory integrals of this type.

For the remainder of this section we will assume that we have fixed the values $\rho = 10$ and $c = 1/2$ and that we are working with the integral $I(x)$ as defined in (5.2) . The behaviour of the real part of integrand when $x = 20$ is pictured in Figure 5.2. We will continue to work on a domain of numerical integration of $[-L, L]$ which is divided into $2N$ intervals of size Δ. To facilitate the discussion, let us denote the interval $[(n-1)\Delta - L, n\Delta - L]$ by $[u_{n-1}, u_n]$ for all $n = 1, \cdots, 2N$, and denote the open and half-open intervals analogously. Further, we define

$$g(u;x) := f(c+iu)e^{iux}, \quad g^*(u;x) := \mathrm{Re}(g(u;x)), \quad \text{and} \quad I^*(x) := \int_{\mathbb{R}} g^*(u;x)\mathrm{d}u.$$

Our aim is not to make rigorous arguments, but rather to present some graphical evidence. In particular, we will demonstrate how various schemes approximate the real part of the integrand and integral, namely $g^*(u;x)$ and $I^*(x)$.

The right endpoint scheme approximates $g(u;x)$ by a "step function" which takes the constant value $g(u_n;x)$ on the interval $(u_{n-1},u_n]$. The integral of this step function is the right endpoint approximation of $I(x)$. In Figure 5.3 we show the real part of the step function compared with $g^*(u;x)$ for the case $\Delta = 0.25$ and $x = 20$. We see that unless we make Δ much smaller, this approach will not give a good approximation for $I^*(x)$.

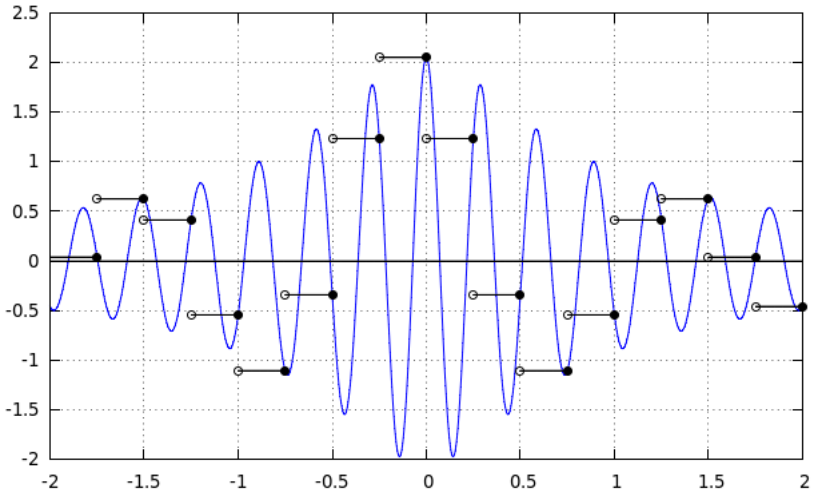


Figure 5.3: The function $g^*(u;20)$ on the interval $[-2,2]$ is graphed in blue. The right endpoint approximation of $I^*(x)$ is given by the area under the approximating step function; the step function is drawn in black. The value of Δ is set to 0.25.

A popular generalization of the right endpoint scheme is the Newton-Cotes scheme. For this scheme, rather than approximating the integrand over each interval by a constant, it is approximated over several intervals by a polynomial. For example, in the most popular variant of the Newton-Cotes scheme, known as Simpson's rule, the integrand is approximated over two adjacent intervals by the Lagrange polynomial. To apply Simpson's rule to our problem, we must approximate $g(u;x)$ over the intervals $j_k := [u_{2k-2},u_{2k}]$, $k = 1,\ldots,N$ by the interpolatory polynomials $p_k(u;x)$, $k = 1,\ldots,N$ which satisfy $p_k(u_{2k-2};x) = g(u_{2k-2};x)$, $p_k(u_{2k-1};x) = g(u_{2k-1};x)$, and $p_k(u_{2k};x) = g(u_{2k};x)$. The integral $\int_{u_{2k-2}}^{u_{2k}} p_k(u;x)\mathrm{d}u$ is the Simpson's rule approximation of $I(x)$ on the interval j_k. Repeating this procedure for the remaining intervals and summing the results gives the ***Simpson's rule***[1] approximation of

[1]Really we are using *composite* Simpson's rule here but, for simplicity, we will refer to both the approximation of $I(x)$ over j_k *and* of $I(x)$ over $[-L,L]$ by the name "Simpson's rule"

$I(x)$ over $[-L, L]$. We observe from Figure 5.4 that our application of Simpson's rule will

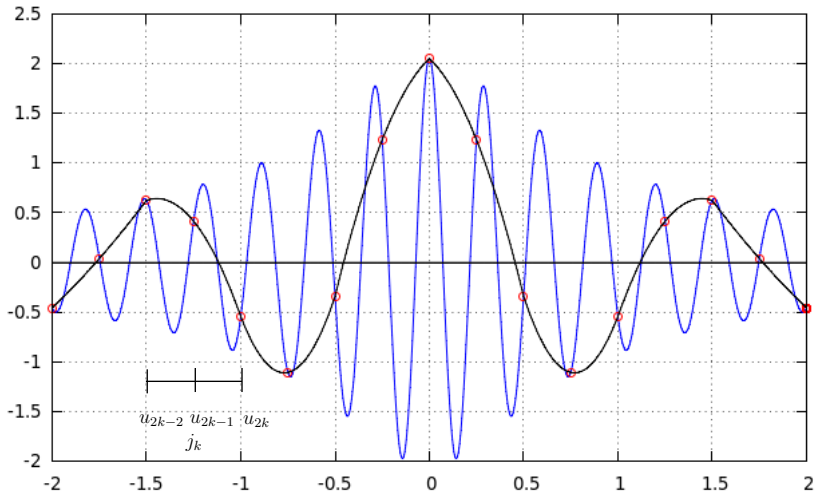


Figure 5.4: The function $g^*(u; 20)$ on the interval $[-2, 2]$ is graphed in blue. On each interval j_k the polynomial $p_k^*(u; 20)$ is drawn in black. The values $g^*(u_n; 20)$ are marked with red circles and Δ is set to 0.25.

give a very bad approximation of $I^*(x)$. The reason, is that on each interval j_k the real polynomial $p_k^*(u; x) := \mathrm{Re}(p_k(u; x))$ approximates $g^*(u; x)$ very poorly. This occurs because $p_k^*(u; x)$ has degree 2, and $g^*(u; x)$ has at least three local extrema in j_k. However, if x is smaller, then $g^*(u; x)$ oscillates more slowly, and our application of Simpson's rule performs reasonably well, as we can see from Figure 5.5.

We conclude that to successfully implement a Newton-Cotes style scheme for $g(u; x)$ we need to adapt the scheme to change with x. That is, as x grows we need to decrease the size of Δ and increase the degree of our approximating polynomial. Alternatively, we can use a small Δ and suitably high degree that works for large enough x; this will lead to increased computational effort.

The appeal of the right endpoint scheme and the Newton-Cotes schemes is that we approximate $g(u; x)$ by a function which is easy to integrate, namely a constant function or a polynomial. Filon [44] recognized that we can also easily integrate

$$\ell(u)e^{iux} = \ell(u)\cos(ux) + i\ell(u)\sin(ux), \quad u \in \mathbb{R}$$

where $\ell(u)$ is a polynomial. Filon's method for $g(u; x)$ is to approximate $f(c + iu)$ by an

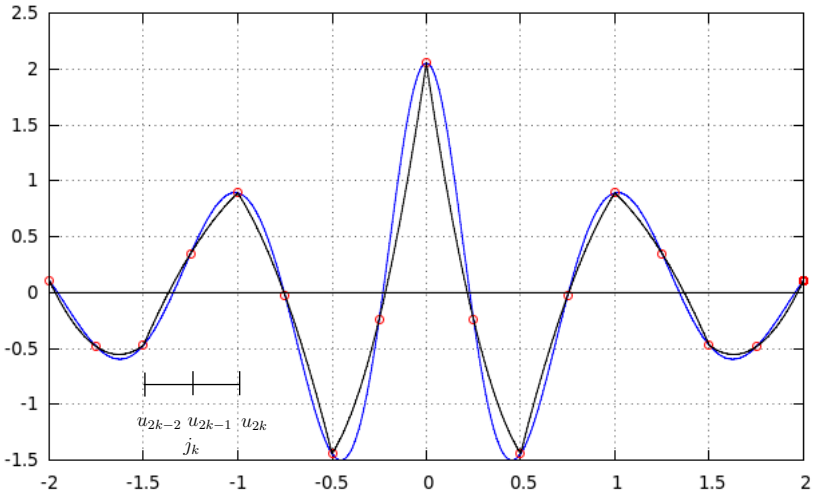


Figure 5.5: The function $g^*(u;5)$ on the interval $[-2,2]$ is graphed in blue. On each interval j_k the polynomial $p_k^*(u;5)$ is drawn in black. The values $g^*(u_n;5)$ are marked with red circles and Δ is set to 0.25.

interpolatory polynomial $\ell_k(u)$ over the interval j_k and approximate $g(u;x)$ by the function $h_k(u;x) := \ell_k(u)e^{iux}$. Integrating $h_k(u;x)$ over j_k and summing the results over k gives the ***Filon's method*** approximation for $I(x)$ over $[-L,L]$. Of course, just as in the Newton-Cotes approximation, j_k does not have to be made up of just two adjacent intervals and the polynomial $\ell_k(u)$ does not have to be the interpolatory polynomial which passes through only three points. However, for our demonstration we assume this is the case.

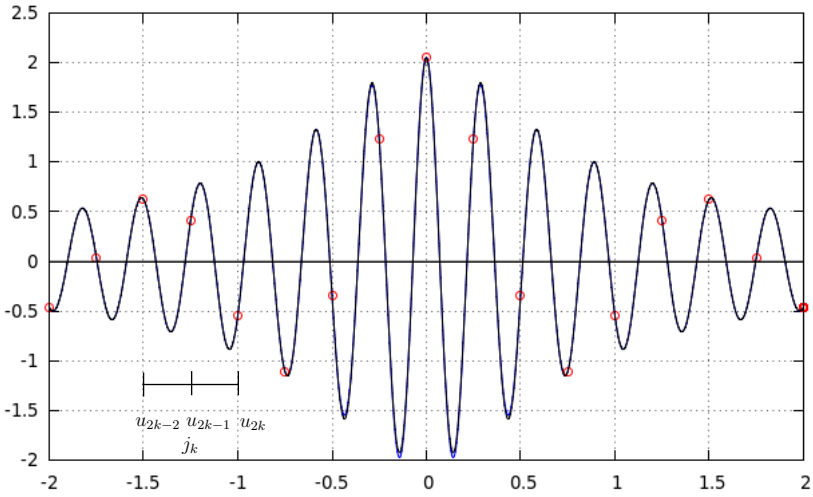


Figure 5.6: The function $g^*(u;20)$ on the interval $[-2,2]$ is graphed in blue. On each interval j_k the function $h_k^*(u;20)$ is drawn in black. The values $g^*(u_n;20)$ are marked with red circles and Δ is set to 0.25.

In Figure 5.6 we see how well $h_k^*(u;x)$ approximates the integrand $g^*(u;x)$. In fact, we can barely distinguish between the approximating curve and the function $g^*(u;x)$. The reason Filon's method works so well, is that we are capturing the x-dependent oscillatory behavior of $g(u;x)$ in our approximation. Therefore, even as x grows, and the frequency of the oscillations increases, our approximation remains accurate without decreasing Δ or increasing the degree of $\ell_k(u)$.

Remark

We note that we do not have to use evenly spaced increments in our discretization scheme in order to implement any of the methods discussed above. Near the origin, the integrand in our example has a high amplitude, but as u grows the function decays. Therefore, it makes sense to use more discretization points near the origin, and widen the intervals as u grows. The reader may assume that we typically use such a strategy in our application of Filon's method in later chapters.

5.2.3 The solutions of $\psi(z) = q$

In this section, we motivate our discussion with an example from finance. Recall from Section 4.2.4 that we may write the Laplace-Mellin transform $\Phi(z,q)$ of the price of an Asian option in terms of the Mellin transform $\mathcal{M}(I_q, z)$ of the exponential functional $I_q(X)$. Specifically, the function $\Phi(z,q)$ takes the form

$$\Phi(z,q) = \frac{\mathcal{M}(I_q, z+2)}{z(z+1)}. \tag{5.6}$$

To recover the option price, our first task is to invert the Mellin transform, which we may do, for a suitably chosen $c \in \mathbb{R}$, as follows:

$$h_a(k,q) = \frac{k^{-c}}{2\pi} \int_{\mathbb{R}} \Phi(c+iu, q) e^{iu\log(k)} \mathrm{d}u. \tag{5.7}$$

Similarly, for a suitable q_0 we invert the Laplace transform to recover the function $f_a(k,t)$, which is essentially the price of the option (see (4.2) and (4.3)):

$$f_a(k,t) = \frac{e^{q_0 t}}{2\pi} \int_{\mathbb{R}} \frac{h_a(k, q_0+iu)}{q_0+iu} e^{iut} \mathrm{d}u = \frac{2e^{q_0 t}}{\pi} \int_{\mathbb{R}^+} \mathrm{Re}\left(\frac{h_a(k, q_0+iu)}{q_0+iu} \right) \cos(ut) \mathrm{d}u. \tag{5.8}$$

The last equality on the right-hand side in (5.8) results from the fact that $f_a(k,t)$ is a real-valued function, which means that we may rewrite the inversion formula in terms of the cosine transform.

In Chapter 6 we show that when X is a meromorphic process $\mathcal{M}(I_q, z)$ is parameterized by the poles of the Laplace exponent $\psi(z)$ and the solutions of $\psi(z) = q$. This means that if our log-stock price is a modeled by a meromorphic process, then we need to find solutions of the equation $\psi(z) = q$ (recall that the poles of $\psi(z)$ are known) in oder to compute $f_a(k,t)$. Of course, from our discussion in Section 4.2.3 we know the situation is essentially the same for hyper-exponential processes. The difference between the two is that there are only finitely many solutions in the hyper-exponential case.

We outline here two problems, and the appropriate remedies, to finding the solutions of $\psi(z) = q$ when X is a meromorphic process. The latter of the two problems/remedies applies also to hyper-exponential processes. Since we only use processes from the β and θ-classes in this thesis we will restrict our discussion to those. Also, we assume that any θ-process has a closed form expression for $\psi(z)$.

Recall that if q is real, then all of the solutions of $\psi(z) = q$ will also be real and they will interlace with the poles (see (2.16)). If X is a process in the β-class (resp. θ-class) then the nth positive pole is given by $\rho = \beta(\alpha + n - 1)$ (resp. $\rho = \alpha + n^2\beta$) for some constants β, $\alpha > 0$; an identical formula holds for the negative poles in both cases. Finding the small solutions of $\psi(z) = q$ is therefore a trivial numerical exercise since the solutions are located on narrow intervals between successive poles. However, for large solutions, the intervals between the poles may no longer be narrow. This means that the starting interval for the numerical method (e.g. bisection or Newton's method) is wide, leading to a computationally expensive algorithm. Therefore, the first problem is finding large solutions of $\psi(z) = q$.

The second problem arises from expression (5.8). We see from (5.8) that we will have to evaluate $h_a(k,q)$, and therefore – working backwards through equations (5.7) and (5.6) – also $\mathcal{M}(I_q, z)$, for complex q. This is problematic from a theoretical standpoint; we defer the discussion on this matter to Chapter 6. Here we address the practical issue of finding solutions to $\psi(z) = q$ when $q = q_0 + iu$, for q_0, $u > 0$.

Large solutions

The issue of large solutions is resolved for most processes in the β and θ-classes in papers [65] and [66]. In these articles Kuznetsov derives asymptotic expansions of the large solutions for all processes in the β-class and key processes in the θ-class. The first terms of these expansions can be used as a good initial guesses for any numerical algorithms; providing a good starting point increases the rate of convergence of a numerical solution. Rather than listing the asymptotic expansions for the various cases here, we will give only the results for β-processes with Gaussian component $\sigma \neq 0$; the remaining expansions can be found in [65,66]. For the following theorem, and for the remainder of this section we revert to the usual notation of denoting the positive solutions of $\psi(z) = q$ by $\{\zeta_n(q)\}_{n\geq 1}$ and the negative solutions by $\{-\hat{\zeta}_n(q)\}_{n\geq 1}$. We will usually suppress the q-dependence and write just ζ_n and $\hat{\zeta}_n$. For the parameters of the processes we use the same notation as in Section 2.2.3.

Theorem 21 (Theorem 10 in [65]). *Let X be a β-process with parameters c_i, α_i, $\beta_i > 0$, $\lambda_i \in (0,3)/\{1,2\}$, $i \in \{1,2\}$, and Gaussian component $\sigma \neq 0$. Then the solutions of $\psi(z) = q$ obey*

$$\hat{\zeta}_{n+1} = -\beta_2(n+\alpha_2) - \frac{2c_2}{\sigma^2\beta_2^2\Gamma(\lambda_2)}(n+\alpha_2)^{\lambda_2-3} + O(n^{\lambda_2-3-\varepsilon}), \quad n \to +\infty, \quad \textit{and}$$
$$\zeta_{n+1} = \beta_1(n+\alpha_1) + \frac{2c_1}{\sigma^1\beta_1^2\Gamma(\lambda_1)}(n+\alpha_1)^{\lambda_1-3} + O(n^{\lambda_1-3-\varepsilon}), \quad n \to +\infty,$$

for some $\varepsilon > 0$.

Solutions for $q \in \mathbb{C}$

For complex q of the form $q = q_0 + iu$, q_0, $u > 0$, we lose the important interlacing property (2.13), (2.16), and therefore the approximate location of the solutions. Also, we may no longer assume the solutions will be real; in fact, we will see that they are not. Three other important questions arise: Will the solutions continue to be of order one or can they have higher order? Will there still be countably many solutions? Most importantly, how can we find the solutions of $\psi(z) = q$, and can we do it efficiently?

These questions are more or less answered in [65], Section 5. First, there is strong numerical evidence to suggest that the solutions remain of order one. Experimentation with the β-class in [65] shows that the solutions when u is small are complex and very near the real solutions of $\psi(z) = q_0$. As u grows all but one of the complex solutions converges to a nearby pole, and avoids any other solutions (i.e. there are no solutions of order 2 or greater).

One of the solutions, either ζ_1 or $-\hat{\zeta}_1$, will grow to ∞, but will avoid the other solutions. Kuznetsov states that we may also show rigorously that for each solution of $\psi(z) = q_0$ we may identify a unique solution of $\psi(z) = q$, and the set of solutions derived in this way comprise all of the solutions of $\psi(z) = q$. That is, there will be no new solutions which we cannot anticipate, and the number of solutions remains countable.

To find a solution ζ of $\psi(z) = q_0 + iu$ we may use the following method outlined in [65]. We may view ζ as an implicitly defined function of u which satisfies,

$$q_0 + iu - \psi(\zeta(u)) = 0, \quad \zeta(0) = \zeta_0,$$

where ζ_0 is the solution of $\psi(z) = q_0$. Differentiating each side with respect to u gives the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}u}\zeta(u) = \frac{i}{\psi'(\zeta(u))},$$

with initial condition $\zeta(0) = \zeta_0$. Such an equation can be solved nicely by a numerical scheme like the midpoint method. If we choose the midpoint method, the algorithm for finding $\zeta(u)$ proceeds along the following lines.

(i) Divide the interval $[0, u]$ into J sections of equal size Δ. Let us denote the endpoint of the j-th interval by u_j, $j = 1, \cdots, J$.

(ii) Suppose we have found $\zeta(u_j)$. Then, we approximate $\zeta(u_{j+1})$ by the midpoint method

$$\zeta(u_{j+1}) = \zeta(u_j) + \frac{i\Delta}{\psi'(\zeta(u_j) + i\Delta/2\psi'(\zeta(u_j))}.$$

(iii) At each step, recalling that $\zeta(u_j)$ is an approximate solution of $\psi(z) = q_0 + iu_j$, we perform a number of steps of Newton's method with $\zeta(u_j)$ as the initial guess. We replace $\zeta(u_j)$ by the outcome of this algorithm and proceed with the next step of the midpoint method.

Remark

From previous chapters we recall that there are other instances and processes for which we might wish to find solutions to the equation, $\psi(z) = q$. For example in Section 3.2.3 we showed that the Wiener-Hopf factors of processes with positive jumps of rational transform, and the Wiener-Hopf factors of processes with bounded positive jumps, are expressed in

terms of these solutions. In Section 4.2.3 we showed the same for the Mellin transform of processes with jumps of rational transform. In these cases, we do not have as much information about the location of the solutions as we do for meromorphic processes or hyper-exponential processes. However, Theorem 2 gives us information about the number and order of the solutions for processes with jumps of rational transform, and there is similar information for processes with positive jumps of rational transform in [83]. For processes with with bounded positive jumps, there is information about the location of the solutions in [73].

In all of the above mentioned cases we are faced with the problem of finding solutions of an equation in the complex plane. A popular approach to solve this problem is based on Cauchy's argument principle. For a general introduction to the numerical applications of the argument principle see Section 16.5 in [1]. For an application of the argument principle to processes with bounded positive jumps consult Section 4.1 in [73].

5.3 Rational approximations

In this section we gather some important facts about Stieltjes and Pick functions and rational approximations of these functions. Such functions are important because we may write the Laplace exponent $\psi(z)$ of (almost) any completely monotone Lévy process in terms of either Stieltjes functions or Pick functions (see Theorem 23). In Chapter 8 we focus on completely monotone processes, and ask: Can we find a rational approximation $\tilde{\psi}(z)$ of $\psi(z)$ such that $\tilde{\psi}(z)$ is the Laplace exponent of a hyper-exponential process? It turns out that the answer is "yes" and the reason is that certain rational approximations of Stieltjes and Pick functions have certain special properties.

In Section 5.3.1 we first give the necessary definitions and then demonstrate the connection between Pick functions, Stieltjes functions, and the Laplace exponent of a completely monotone process. Then, in Section 5.3.2 we study rational approximations of Pick and Stieltjes functions. We pay special attention to Padé approximants and their properties as these are the most important for the work in Chapter 8.

5.3.1 Completely monotone Lévy processes, Pick functions, and Stieltjes functions

First, we recall that a completely monotone Lévy process X has a Lévy density of the form,

$$\pi(x) = \mathbb{I}(x < 0) \int_{\mathbb{R}^-} e^{-ux} \mu(\mathrm{d}u) + \mathbb{I}(x > 0) \int_{\mathbb{R}^+} e^{-ux} \mu(\mathrm{d}u), \tag{5.9}$$

where $\mu(\mathrm{d}u)$ is a measure with support in $\mathbb{R}\backslash\{0\}$ that satisfies the condition

$$\int_{\mathbb{R}\backslash\{0\}} \frac{1}{|u|(1+u)^2} \mu(\mathrm{d}u) < \infty. \tag{5.10}$$

Next we define

$$\begin{aligned}
\rho &:= \sup\left\{c \geq 0 : \int_{\mathbb{R}^+} e^{cx} \pi(x) \mathrm{d}x < \infty\right\} = \sup\left\{u \geq 0 : \mu((0,u)) = 0\right\}, \text{ and} \\
\hat{\rho} &:= \sup\left\{c \geq 0 : \int_{\mathbb{R}^-} e^{-cx} \pi(x) \mathrm{d}x < \infty\right\} = \sup\left\{u \geq 0 : \mu((-u,0)) = 0\right\}.
\end{aligned}$$

It is easy to see that the quantities ρ and $\hat{\rho}$ are strictly positive if, and only if, the Lévy density of X decreases exponentially fast as $|x| \to \infty$. We will denote by $\mathcal{CM}(\hat{\rho}, \rho)$ the class of Lévy processes with completely monotone jumps with parameters ρ and $\hat{\rho}$.

For $\alpha \leq \beta$ denote by $P(\alpha, \beta)$ the class of functions $\{f(z)\}$ which are analytic in $\mathbb{C}\backslash((-\infty, \alpha] \cup [\beta, \infty))$ and satisfy: $f(\mathbb{C}^+) \subset \mathbb{C}^+$, and $f(\bar{z}) = \overline{f(z)}$ for $z \in \mathbb{C}^+$. We will say a function $f(z)$ is a ***Pick function*** if $f(z) \in P(\alpha, \beta)$ for some $\alpha \leq \beta$. From [101] we have the following useful alternative representation of Pick functions.

Theorem 22 (Theorem 1 in [101]). *The function $f(z)$ is in $P(\alpha, \beta)$ if, and only if, for $z \notin (-\infty, \alpha] \cup [\beta, \infty)$,*

$$f(z) = a + \sigma z + \int_{\mathbb{R}} \left(\frac{1}{u - z} - \frac{u}{1 + u^2}\right) \mu(\mathrm{d}u), \tag{5.11}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\mu(\mathrm{d}u)$ puts no mass on (α, β), and $\int_{\mathbb{R}} (1 + u^2)^{-1} \mu(\mathrm{d}u) < \infty$.

Example: z^α

Consider the function z^α where $0 < \alpha < 1$. Clearly, $z^\alpha \in P(0, \infty)$; we would like to find the (5.11) representation of z^α which we claim has the form

$$\mu(\mathrm{d}u) := \mathbb{I}(u < 0)\frac{|u|^\alpha \sin(\alpha\pi)}{\pi}\mathrm{d}u, \quad a := \int_{\mathbb{R}} \frac{\mu(-\mathrm{d}u)}{u(1+u^2)}, \quad \text{and} \quad \sigma = 0.$$

Proving this is largely a matter of algebraic manipulation. Making a change of variables, and factoring out $(1+u^2)^{-1}$ gives

$$\int_{\mathbb{R}} \left(\frac{1}{u-z} - \frac{u}{1+u^2}\right)\mu(\mathrm{d}u) = \int_{\mathbb{R}} \left(\frac{zu-1}{z+u}\right)\frac{\mu(-\mathrm{d}u)}{1+u^2}.$$

Some further algebra,

$$\int_{\mathbb{R}} \left(\frac{zu-1}{z+u}\right)\frac{\mu(-\mathrm{d}u)}{1+u^2} = \int_{\mathbb{R}} \frac{z}{z+u}\left(u+\frac{1}{u}\right)\frac{\mu(-\mathrm{d}u)}{1+u^2} - \int_{\mathbb{R}} \frac{1}{u}\frac{\mu(-\mathrm{d}u)}{(1+u^2)},$$

demonstrates that the problem reduces to showing that

$$z^\alpha = \frac{\sin(\alpha\pi)}{\pi}\int_{\mathbb{R}^+} \frac{z}{z+u}u^{\alpha-1}\mathrm{d}u. \tag{5.12}$$

Once we recall that $\pi^{-1}\sin(\alpha\pi) = (\Gamma(1-\alpha)\Gamma(\alpha))^{-1}$ and $(z+u)^{-1} = \int_{\mathbb{R}^+}\exp(-t(z+u))\mathrm{d}t$, for $\mathrm{Re}(z)$, $u > 0$, equality (5.12) follows easily for $\mathrm{Re}(z) > 0$. Since the right-hand side of (5.12) is an analytic function of z on $\mathbb{C}\backslash(-\infty, 0]$ the equality extends to all $z \in \mathbb{C}\backslash(-\infty, 0]$ by analytic continuation.

A ***Stieltes function*** is defined by the Stieltjes-integral representation,

$$f(z) := \int_{\bar{\mathbb{R}}^+} \frac{\nu(\mathrm{d}u)}{1+zu}, \quad z \in \mathbb{C}\backslash(-\infty, 0],$$

where $\nu(\mathrm{d}u)$ is a positive measure on $\bar{\mathbb{R}}^+$ whose support has infinitely many different values, and which has finite moments

$$m_j := \int_{\bar{\mathbb{R}}^+} u^j \nu(\mathrm{d}u).$$

Formally, we may also express $f(z)$ as a ***Stieltjes series***, which may converge only at 0, and has the following form:

$$f(z) = \sum_{j=0}^{\infty} (-z)^j m_j. \tag{5.13}$$

It is easy to see that the above series converges for $|z| < R$ if and only if $\text{supp}(\nu) \subseteq [0, 1/R]$. In this case we will call $f(z)$ a ***Stieltjes function (or a Stieltjes series) with the radius of convergence*** R. For such functions, the domain of definition extends to all $z \in \mathbb{C}\backslash(-\infty, -R]$.

Example: $z^{-1}\log(1+z)$ (from pg. 161 in [9])

Let us show that the function, $z^{-1}\log(1+z)$, is a Stieltjes function/series with radius of convergence, $R = 1$. Expanding as a Taylor series at $z = 0$ gives

$$\frac{\log(1+z)}{z} = 1 - \frac{1}{2} + \frac{1}{3}z^2 \pm \ldots \pm \frac{1}{j+1}z^j \pm \ldots, \quad |z| < 1.$$

We observe that $(j+1)^{-1} = \int_0^1 u^j \mathrm{d}u$, so that for $|z| < 1$ we obtain

$$\frac{\log(1+z)}{z} = \int_0^1 \frac{\mathrm{d}u}{1+zu}.$$

Then, by analytic continuation we may extend the above equality to all of $\mathbb{C}\backslash(-\infty, -1]$. Therefore, $z^{-1}\log(1+z)$ is a Stieltjes function with measure $\nu(\mathrm{d}u) = \mathbb{I}(0 < u < 1)\mathrm{d}u$, and radius of convergence $R = 1$.

As we have alluded to in the Introduction there is a connection between the Laplace exponents of completely monotone processes, Stieltjes functions and Pick functions. By relatively simple transformations, we can represent the Laplace exponent $\psi(z)$ either in terms of a Stieltjes or Pick function. The precise statement of the connection is given in Theorem 23.

Theorem 23. *Assume X is a Lévy process with Laplace exponent $\psi(z)$ and $\hat{\rho}$, $\rho > 0$. The following assertions are equivalent:*

(i) $X \in \mathcal{CM}(\hat{\rho}, \rho)$.

(ii) $\frac{\psi(z)}{z} \in P(-\hat{\rho}, \rho)$.

(iii)

$$\psi(z) = az + \frac{\sigma^2}{2}z^2 + \frac{z^2}{1+\frac{z}{\hat{\rho}}}g\left(-\frac{z}{1+\frac{z}{\hat{\rho}}}\right),$$

where a, $\sigma \in \mathbb{R}$, and $g(z)$ is a Stieltjes function with radius of convergence $R = (1/\rho + 1/\hat{\rho})^{-1}$.

We will give our own proof of $(i) \Leftrightarrow (ii)$, although this is essentially just a reformulation of Theorem 22. We delay the proof of $(i) \Leftrightarrow (iii)$ until Chapter 8 (page 153). Similar ideas, with slightly different definitions for Stieltjes and Pick functions, can also be found in [104].

Proof. To prove $(i) \Rightarrow (ii)$ we define the cut-off function $g(x) = |x|^{-1}\left(1 - e^{-|x|}\right)$ and use the generating triple $(a, \sigma^2, \pi)_{h\equiv g}$. We then write the Laplace exponent of our completely monotone process as

$$\psi(z) = az + \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - \operatorname{sgn}(x)\left(1 - e^{-|x|}\right)z\right)\pi(x)\mathrm{d}x, \tag{5.14}$$

where $\pi(x)$ is a Lévy measure of the form (5.9). Then using (5.9), (5.14), and applying Fubini's theorem, we find that the $z^{-1}\psi(z)$ can be expressed as

$$\frac{\psi(z)}{z} = \tilde{a} + \frac{\sigma^2}{2}z + \int_{\mathbb{R}} \left(\frac{1}{u-z} - \frac{u}{1+u^2}\right)\nu(\mathrm{d}u), \tag{5.15}$$

where we have defined

$$\nu(\mathrm{d}u) := \frac{\mu(\mathrm{d}u)}{|u|},$$

and

$$\tilde{a} := a + \int_{\mathbb{R}} \left(\frac{u}{1+u^2} - \mathbb{I}(u>0)\frac{1}{u+1} - \mathbb{I}(u<0)\frac{1}{u-1}\right)\nu(\mathrm{d}u).$$

This completes the the proof of $(i) \Rightarrow (ii)$. For the proof of the converse we start with the expression (5.15) and simply reverse the steps. □

We remark that in the proof, we did not use the fact that ρ, $\hat{\rho} > 0$, or equivalently that the Lévy density has exponential tails. However, this assumption assures that the function $z^{-1}\psi(z)$ is analytic for an interval on the real line, which is desirable for our further discussion. For our proof of $(i) \Leftrightarrow (iii)$ the assumption ρ, $\hat{\rho} > 0$ is necessary.

5.3.2 Rational approximation of Pick and Stieltjes functions

Given a function $f : \mathbb{C} \to \mathbb{C}$, k points $z_1, \ldots, z_k$, and equally many non-negative integers $\beta_1, \ldots, \beta_k$, consider the following interpolation problem: Denote $N := \sum_{1 \leq i \leq k} (\beta_i + 1)$ and find a rational function $g(z)$ whose numerator and denominator have degree at most $N/2$, which satisfies

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j} g(z)\bigg|_{z=z_i} = \frac{\mathrm{d}^j}{\mathrm{d}z^j} f(z)\bigg|_{z=z_i}, \quad 1 \leq i \leq k,\, 0 \leq j \leq \beta_i.$$

This problem is known as a ***Cauchy interpolation problem (of degree N)***; we will call a solution of the problem a ***Cauchy interpolant (of degree N)***. It is quite clear that a Cauchy interpolation problem does not always have a solution. To see this, we consider a simple example (from pg. 19 in [9]) where we set $k = 1$, $z_1 = 0$, $\beta_1 = 2$ (and therefore $N = 3$), and we wish to interpolate the function $f(z) = 1 + z^2$. Let us assume that a Cauchy interpolant $g(z)$ exists, in which case we may write

$$g(z) = \frac{P(z)}{Q(z)}, \quad P(z) = p_0 + p_1 z, \quad Q(z) = q_0 + q_1 z.$$

Then, we must have

$$\frac{p_0 + p_1 z}{q_0 + q_1 z} = 1 + z^2 + O(z^3). \tag{5.16}$$

Cross-multiplying and equating coefficients in (5.16) shows that $p_0 = q_0 = 0$, and $p_1 = q_1$, so that $g(z) = 1$. Clearly, the second derivative of 1 does not match the second derivative of $1 + z^2$ at 0, so we have a contradiction.

The previous example introduces a special variant of the Cauchy interpolation problem in which we interpolate only at one point $z_1 = a$. Since this type of interpolation is the most important for this work, let us define it formally here. Let $f(z)$ be a function with a power series representation $f(z) = \sum_{i \geq 0} c_i (z - a)^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P_m) \leq m$, $\deg(Q_n) \leq n$, $Q_n(a) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = f(z) + O((z - a)^{m+n+1}), \quad z \to a,$$

then we say that $f^{[m/n]}(z) := P_m(z)/Q_n(z)$ is the $[m/n]$ ***Padé approximant of*** $f(z)$ ***at point*** a. For the special case $a = 0$ we will say simply that $f^{[m/n]}(z)$ is the $[m/n]$ ***Padé approximant of*** $f(z)$. If we write simply $f^{[m/n]}(z)$ without reference to a, the reader should

assume that $a = 0$.

We notice two slight differences between the definition of a Padé approximant and a Cauchy interpolant. First, for a Padé approximant we insist that the constant term in the denominator is equal to 1. Second, we no longer insist that $\deg(P_m)$ and $\deg(Q_n)$ are both less than or equal to $N/2$. The first condition ensures that if $f^{[m/n]}(z)$ exists it is the *unique* rational function which satisfies the conditions of our interpolation problem (see Theorem 1.4.3 in [9]). The second condition simply gives us more freedom in determining a rational approximation; we will see that this is beneficial for our application in Chapter 8.

Now we state some remarkable facts about Cauchy interpolants and Padé approximants of Pick and Stieltjes functions. First let us consider Pick functions.

Theorem 24 (Theorems 3 and 4 in [40])**.** *Let $f(z) \in P(\alpha, \beta)$, such that $f(z)$ is itself not a rational function. Then for points, $z_1, \ldots, z_k$, lying inside the interval (α, β) and for odd N, there exists a unique solution $g(z)$ to the Cauchy interpolation problem of degree N and $g(z) \in P(\alpha, \beta)$.*

From the same source, we also have the following useful lemma concerning rational Pick functions.

Lemma 2 (Discussion, pg. 172 in [40])**.** *Any rational function belonging to $P(\alpha, \beta)$ has simple, real poles in $(-\infty, \alpha] \cup [\beta, \infty)$ which have negative residues.*

Let us put together the information of Theorems 23 and 24 and Lemma 2. Suppose, that we have a completely monotone process X with Laplace exponent $\psi(z)$. Then, for any odd N we may approximate $\psi(z)$ by a rational function

$$\tilde{\psi}(z) = zg(z) = zP(z)/G(z),$$

where $\deg(P) \leq N/2$ and $\deg(Q) \leq N/2$, and $g(z)$ is a Cauchy interpolant of degree N of the Pick function $z^{-1}\psi(z)$. This has the form

$$g(z) = a + \sigma z + \sum_{k=1}^{K} \frac{a_k}{b_k - z} - \sum_{k=1}^{\hat{K}} \frac{\hat{a}_k}{\hat{b}_k + z},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\{-\hat{b}_k\}_{1 \leq k \leq \hat{K}}$ and $\{b_k\}_{1 \leq k \leq K}$ are the negative and positive poles of $g(z)$, $\{-\hat{a}_k\}_{1 \leq k \leq \hat{K}}$ and $\{-a_k\}_{1 \leq k \leq K}$ are the corresponding residues, and $0 < K + \hat{K} < N/2 - \delta$.

Here δ is a quantity in $\{0, 1, 2\}$ which depends on the values of a and σ. We see, by comparing with (2.11), that we have approximated $\psi(z)$ by the Laplace exponent $\tilde{\psi}(z)$ of a hyper-exponential Lévy process! All that is required to derive the approximation is to solve a Cauchy interpolation problem for the Pick function $z^{-1}\psi(z)$.

Once we know a solution exists, solving the Cauchy interpolation problem is a matter of solving a system of linear equations (see Chapter 8 in [8] and Section 5.3.3 below for details). In general this is a relatively straightforward task for which we may use standard techniques from numerical analysis. Specific (faster) algorithms also exist if we want to know the value of $g(z)$ at only one point, rather than deriving the coefficients of $P(z)$ and $Q(z)$ (see again Chapter 8 in [8]).

The connection between Pick functions, the Cauchy interpolant, and the Laplace exponents of completely monotone processes is both interesting and useful. We can obtain even more useful results if we focus on the connection between Stieltjes functions and Padé approximants. First, we have the analogue of Theorem 24 and Lemma 2 for Stieltjes functions.

Theorem 25 (Corollary 5.1.1, Theorem 5.2.1 in [9])**.** *If $f(z)$ is a Stieltjes function with radius of convergence $R \geq 0$, then $f^{[m/n]}(z)$ exists provided $m \geq n - 1$. The approximant $f^{[m/n]}(z)$ has simple poles in $(-\infty, -R]$ which have positive residues.*

If, as we did for Pick functions, we consider the information of Theorem 23 and Theorem 25 together we may show another way to approximate the Laplace exponent of a completely monotone Lévy process with the Laplace exponent of a hyper-exponential process, this time via Padé approximants and Stieltjes functions. However, we will see in Chapter 8 that a more natural way to present these ideas is via the Gaussian quadrature, and we save the proof of this fact until then. The major advantage of approaching the approximation problem via Padé approximants of Stieltjes functions is that there exists a well-established theory on the subject which has connections to the Gaussian quadrature, orthogonal polynomials, continued fractions, and the moment problem. Also, we have existing convergence results for Padé approximants of Stieltjes functions. The remainder of this section will be devoted to stating the main theorems which establish the connections to the Gaussian quadrature and orthogonal polynomials, and stating a theorem on the convergence results. Also, we reproduce here several important theorems about transformations of Padé approximants which will be necessary for the work in Chapter 8.

First, let us formally define the Gaussian quadrature and orthogonal polynomials. Consider a finite positive measure $\nu(\mathrm{d}x)$ on an interval $[a, b]$ whose support has infinitely many

different values. The ***Gaussian quadrature (of order* n*)*** is the measure $\tilde{\nu}(\mathrm{d}x)$ which is supported on n points inside $[a,b]$, and matches the first $2n-1$ moments of $\nu(\mathrm{d}x)$. The ***weights*** $\{w_i\}_{1\leq i\leq n}$ and the ***nodes*** $\{x_i\}_{1\leq i\leq n}$ of $\tilde{\nu}(\mathrm{d}x)$ are uniquely defined by equations,

$$\int_{[a,b]} x^k \nu(\mathrm{d}x) = \sum_{i=1}^{n} x_i^k w_i, \quad k = 0, 1, \ldots, 2n-1.$$

For a proof of the existence and uniqueness of the Gaussian quadrature see Theorem 3.4.1 in [109].

Orthogonal polynomials with respect to the measure $\nu(\mathrm{d}x)$ are a set of polynomials $\{p_n(x)\}_{n\geq 0}$ which satisfy: $\deg(p_n) = n$, and

$$\int_{[a,b]} p_n(x) x^k \nu(dx) = 0, \quad k = 0, 1, \ldots, n-1.$$

See Theorem 2.1.1 in [109] for a proof that at least one set of orthogonal polynomials exists. To ensure that the set is uniquely defined we can employ a number of normalizations; for example, if we insist that the coefficient a_n of x^n in the polynomial $p_n(x)$ is strictly positive and that,

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = \delta_{m,n},$$

then the set $\{p_n(x)\}_{n\geq 0}$ is uniquely determined. However, we will not concern ourselves with uniqueness in this dissertation. For us, it is enough to know that orthogonal polynomials can be defined uniquely for any $\nu(\mathrm{d}x)$ up to multiplication by a constant. That is, if both $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ represent orthogonal polynomials with respect to $\nu(\mathrm{d}x)$ then there exist $d_n \in \mathbb{R}$ such that $p_n(x) = d_n q_n(x)$ for all $x \in [a,b]$ and $n \geq 0$ (see the Corollary on pg. 9 in [32]). Therefore, we will occasionally write *the* orthogonal polynomials, although uniqueness should be understood in the sense we have just described.

Example: The Jacobi polynomials (Chapter IV in [109])

Consider the measure

$$\nu(\mathrm{d}x) = \mathbb{I}(|x| \leq 1)(1-x)^\alpha (1+x)^\beta \mathrm{d}x, \quad \alpha, \beta > -1.$$

We can show that the polynomials $\{p_n(x)\}_{n\geq 0}$ which are defined as

$$p_n(x) := \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{i=0}^{n} \binom{n}{i} \frac{\Gamma(\alpha+\beta+n+i+1)}{\Gamma(\alpha+i+1)} \left(\frac{z-1}{2}\right)^i,$$

are orthogonal polynomials with respect to $\nu(\mathrm{d}x)$, and that

$$(p_n, p_m)_\nu = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \delta_{m,n}.$$

These polynomials are commonly referred to as the ***Jacobi polynomials***, and we will employ the notation $P_n^{(\alpha,\beta)}(x)$ to denote the nth Jacobi polynomial.

In the previous example we see that the measure $\nu(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure, and has density function $w(x) = (1-x)^\alpha(1+x)^\beta$. This is the case for many classical orthogonal polynomials, and the density $w(x)$ is called the ***weight function*** . When $w(x)$ is an even function defined on a symmetric interval about the origin, as is the case of the Jacobi polynomials, we may shift the domain of the orthogonal polynomials with the linear transformation $x \mapsto k^{-1}(x-\ell)$ where k, $\ell \in \mathbb{R}$ and $k \neq 0$. This mapping carries the original interval $[-a, a]$ into a new interval $[-k^{-1}(a+\ell), k^{-1}(a-\ell)]$ on which we can define the ***shifted orthogonal polynomials*** by

$$\tilde{p}_n(y) := (\mathrm{sign}(k))^n |k|^{1/2} p_n(k)(ky+\ell), \quad n \geq 0.$$

The shifted orthogonal polynomials are the orthogonal polynomials on $[-k^{-1}(a+\ell), k^{-1}(a-\ell)]$ with respect to the measure $\tilde{\nu}(\mathrm{d}y) = w(ky+\ell)\mathrm{d}y$ (see pg. 29 in [109]). The most common application of shifting the domain of definition in this way, and the most important for this dissertation, is to shift the Jacobi polynomials from the interval $[-1, 1]$ to the interval $[0, 1]$.

Now let us state two theorems which tie these ideas together. The first states a well-known connection between orthogonal polynomials and the Gaussian quadrature. The second is less well known, and shows the connection between the Gaussian quadrature, orthogonal polynomials, and the Padé approximant of a Stieltjes function.

Theorem 26 (Theorems 3.4.1 and 3.4.2 in [109]). *Let $\{x_j\}_{1\leq j\leq n}$, and $\{w_j\}_{1\leq j\leq n}$ be the nodes and the weights of the Gaussian quadrature of order n with respect to measure $\nu(\mathrm{d}x)$ and let $\{p_n(x)\}_{n\geq 0}$ be the orthogonal polynomials with respect to $\nu(\mathrm{d}x)$. Then, the nodes $\{x_j\}_{1\leq j\leq n}$*

are given by the zeros of the polynomial $p_n(x)$*, and the weights are given by*

$$w_j = \frac{a_n}{a_{n-1}} \frac{(p_{n-1}, p_{n-1})_\nu}{p_{n-1}(x_j)p'_n(x_j)}, \tag{5.17}$$

where a_k *is the coefficient of* x^k *in* $p_k(x)$*.*

Theorem 27 (Theorems 2.2 and 3.1 in [5])**.** *Consider a Stieltjes function*

$$f(z) = \int_{[0,a]} \frac{\nu(\mathrm{d}x)}{1+xz}.$$

Then

$$f^{[n-1/n]}(z) = \frac{(-z)^{n-1}q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1+x_i z}, \tag{5.18}$$

where $\{x_i\}_{1\le i\le n}$ *and* $\{w_i\}_{1\le i\le n}$ *are the nodes and weights of the Gaussian quadrature of order* n *with respect to the measure* $\nu(\mathrm{d}x)$*,* $p_n(z)$ *is the* n*th orthogonal polynomial with respect to* $\nu(\mathrm{d}x)$ *and* $q_{n-1}(z)$ *is the* ***associated polynomial of degree*** $n-1$*, defined by*

$$q_{n-1}(z) := \int_{[0,a]} \frac{p_n(z) - p_n(x)}{z-x} \nu(\mathrm{d}x).$$

Theorem 27 shows us that when we are approximating a Stieltjes function we can work interchangeably with the Padé approximant, orthogonal polynomials, or the Gaussian quadrature. This is useful, because although we know that we can always calculate the Padé approximant by solving a system of linear equations (see Section 5.3.3 below) this approach will not give us an explicit formula. Using the well-established theory of orthogonal polynomials, we can occasionally derive such a formula. If we choose to tackle the approximation by means of the Gaussian quadrature, the connection with orthogonal polynomials and Theorem 26 can reduce our computational effort because the location of the zeros of certain families of orthogonal polynomials are well studied.

A natural question that arises when approximating any function is: How good is the approximation? In the example in Section 5.3.3 below, we give a numerical example to show that Padé approximants are excellent at approximating Stieltjes functions. The following Theorem gives a theoretical basis for our numerical results.

Theorem 28 (Theorem 5.4.4 in [9]). *Let $f(z)$ be a Stieltjes series with radius of convergence $R > 0$. Let A be a compact subset of $\mathbb{C} \setminus (-\infty, -R]$. Define δ to be the distance from A to the set $(-\infty, -R]$ and $\rho := R - \delta$. Then there exists a constant $C = C(A)$ such that for all $z \in A$ and all $n \geq 1$ we have*

$$|f(z) - f^{[n-1/n]}(z)| < C \left| \frac{\sqrt{\rho + z} - \sqrt{\rho}}{\sqrt{\rho + z} + \sqrt{\rho}} \right|^{2n}.$$

Another natural question is: If we transform our original function, can we express the approximation of the transformed function in terms of the approximation of the original function? If so, what kind of transformation must we apply to the approximation of the original function? In the three theorems below, we see that Padé approximants are well behaved under simple transformations. We will need these results in Chapter 8.

Theorem 29. *(Theorem 1.5.2 in [9]) Given a formal series $f(z) = \sum_{i \geq 0} c_i z^i$ and $a \neq 0$ we define $w = w(z) := az/(1 + bz)$ and $g(w) := f(z)$. If the Padé approximant $f^{[n/n]}(z)$ exists, then $g^{[n/n]}(w) = f^{[n/n]}(z)$.*

Theorem 30. *(Theorem 1.5.3 in [9]) Given a formal series $f(z) = \sum_{i \geq 0} c_i z^i$ we define $g(z) := (a + bf(z))/(c + df(z))$. If $c + df(0) \neq 0$ and the Padé approximant $f^{[n/n]}(z)$ exists, then*

$$g^{[n/n]}(z) = \frac{a + bf^{[n/n]}(z)}{c + df^{[n/n]}(z)}.$$

Theorem 31. *(Theorem 1.5.4 in [9]) Assume that $k \geq 1$ and n, m are integers such that $n - k \geq m - 1$. Given a formal series $f(z) = \sum_{i \geq 0} c_i z^i$ we define*

$$g(z) := \left(f(z) - \sum_{i=0}^{k-1} c_i z^i \right) z^{-k}.$$

Then,

$$g^{[n-k/m]}(z) = \left(f^{[n/m]}(z) - \sum_{i=0}^{k-1} c_i z^i \right) z^{-k},$$

provided either Padé approximant exists.

5.3.3 Numerical considerations and an example

In this section we briefly discuss how to compute the coefficients of Padé approximants directly, i.e. without invoking connections to orthogonal polynomials or the Gaussian quadrature. At the end of the section we calculate the $[n/n]$ Padé approximant of the function $\psi(z) = -\log(1-z)$ using this technique. We will see that for any n the approximant is the Laplace exponent of a hyper-exponential process, and that the approximation is very accurate.

Consider a function $f(z)$ given by a formal series expansion $f(z) = \sum_{i\geq 0} c_i z^i$. Then the Padé approximant $f^{[m/n]}(z) = P_m(z)/Q_n(z)$ with $m \geq n$ can be found as follows (provided it exists). First, we solve the system of n linear equations

$$\begin{pmatrix} c_{m-n+1} & c_{m-n+2} & c_{m-n+3} & \cdots & c_m \\ c_{m-n+2} & c_{m-n+3} & c_{m-n+4} & \cdots & c_{m+1} \\ c_{m-n+3} & c_{m-n+4} & c_{m-n+5} & \cdots & c_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & c_{m+2} & \cdots & c_{m+n-1} \end{pmatrix} \begin{pmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_{m+1} \\ c_{m+2} \\ c_{m+3} \\ \vdots \\ c_{m+n} \end{pmatrix} \tag{5.19}$$

whose solutions b_i, $1 \leq i \leq n$, give us the coefficients of the denominator $Q_n(z) := 1 + b_1 z + b_2 z^2 + \cdots + b_n z^n$. Then, the coefficients of the numerator $P_m(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ can be calculated recursively:

$$\begin{aligned} a_0 &= c_0, \\ a_1 &= c_1 + b_1 c_0, \\ a_2 &= c_2 + b_1 c_2 + b_2 c_0, \\ &\vdots \\ a_m &= c_m + \sum_{i=1}^{n} b_i c_{m-i}. \end{aligned} \tag{5.20}$$

In practice, when n is even moderately large, the system in (5.19) will have a very large condition number, and solving the system of linear equations (5.19) will likely involve a loss

of accuracy. This can be avoided by using higher precision arithmetic. For the numerical results in this work, when we calculate the Padé approximant by solving a linear system, we use Fortran-90 with the MPFUN90 arbitrary precision package. As we know from Theorem 27 another way to deal with this problem is to use expressions for Padé approximants given in terms of the Gaussian quadrature. There exist several very fast and accurate methods for computing the weights and nodes of the Gaussian quadrature, see [47, 48].

Example: The Padé approximant of $\psi(z) = -\log(1-z)$

We recognize the function $\psi(z) = -\log(1-z)$ as the Laplace exponent of a Gamma subordinator. We would like to approximate $\psi(z)$ by a rational function that is also the Laplace exponent of a hyper-exponential process. To do this, we will use the theory of Padé approximants of Stieltjes functions, and the fact that

$$\psi(z) = -zg(-z), \qquad \text{where} \qquad g(z) = \frac{\log(1+z)}{z}.$$

From Theorems 25, 29, 31, and the fact that $g(z)$ is a Stieltjes function, we know that $\psi^{[n/n]}(z)$ exists and has the form

$$\psi^{[n/n]}(z) = -zg^{[n-1/n]}(-z) = z\sum_{k=1}^{n}\frac{a_k}{b_k - z},$$

where $\{-b_k\}_{1\le k\le n}$ are the poles of $g^{[n-1/n]}(z)$, which we know lie in the interval $(-\infty, -1]$, and $\{a_k\}_{1\le k\le n}$ are the corresponding positive residues. Comparing with (2.11) we see that $\psi^{[n/n]}(z)$ is the Laplace exponent of a hyper-exponential subordinator. We know from Theorem 28 that as n grows, $\psi^{[n/n]}(z)$ converges to $\psi(z)$. From Figure 5.7 we see that the approximation becomes accurate very rapidly. For $n = 10$ we see almost no difference between the curves of $\psi^{[n/n]}(z)$ and $\psi(z)$ on the interval $[-100, 1)$.

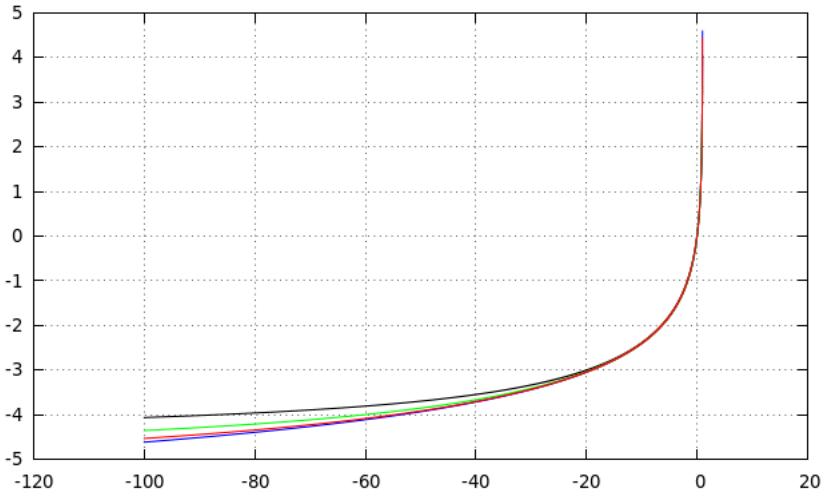


Figure 5.7: The $[n/n]$ Padé approximant of $\psi(z) = -\log(1-z)$ on the interval $[-100, 1)$. The function $\psi(z)$ is plotted in blue, while $\psi^{[n,n]}(z)$ is plotted in black, green, and red for $n = 5, 7$, and 10 respectively.

Part II

New results

Chapter 6

Asian options and meromorphic processes

As we demonstrated in Chapter 4 one method of computing the price of an Asian option in a Lévy driven model is based on the exponential functional of the underlying Lévy process. If we know the distribution of the exponential functional, we can calculate the price of the Asian option via the inverse Mellin-Laplace transform. In this chapter we study the pricing problem in the context of a model driven by a meromorphic Lévy process. We prove that the exponential functional is equal in distribution to an infinite product of independent beta random variables, and its Mellin transform can be expressed as an infinite product of gamma functions. We show that these results lead to an efficient algorithm for computing the price of the Asian option via the inverse Mellin-Laplace transform, and we compare this method with some other techniques.

6.1 Introduction

Our focus in this chapter is to calculate the price of an arithmetic, continuously monitored, fixed strike Asian call option, which we recall from (4.1) is given by

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right]. \tag{6.1}$$

As before, we assume that the expectation in (6.1) is with respect to a risk-neutral measure and that X is a Lévy process. In Chapter 4 we outlined two methods of solution via the exponential functional, one for the traditional Black-Scholes setting, and the other, for

processes for which we can calculate the Mellin transform of the exponential functional (see Sections 4.3.3 and 4.2.4 respectively).

A search of the literature reveals a number of other approaches for calculating $C(A_0, K, T)$. We are, however, unable to find any existing pricing algorithms for cases where the log-stock price is modeled by an infinite activity process with double-sided jumps. This poses a problem, since recent research [3, 29] suggests that stock prices are most realistically modeled by infinite activity processes. Other than by the methods discussed in Chapter 4, we are aware of the following three approaches for calculating $C(A_0, K, T)$: a) approximate the exponential functional I_u by a more tractable random variable; b) define $C(A_0, K, T)$ implicitly as the solution of a backward Kolmogorov equation; and c) use Monte-Carlo methods.

The first approach was used by Milevsky and Posner [87] in the Black-Scholes setting. In this case, the authors approximated I_u by I_∞ in such a way that they matched the first and second moments of the two random variables. This approach is very effective in the Black-Scholes setting since the first and second moments of I_u are known, and I_∞ is a random variable with a known density function.

The second approach is based on the observation that, while the process $Z_t := \int_0^t \exp(X_u)\mathrm{d}u$ is not Markovian, the process

$$\tilde{Z}_t := (x + Z_t)e^{-X_t}$$

does satisfy the Markov property. The process $\tilde{Z}_t$, known as a generalized Ornstein-Uhlenbeck process, is an important and well-studied object (see [84] and the references therein). The fact that $\tilde{Z}$ is a Markov process allows us to rewrite the expectation in (6.1) so that it involves only $\tilde{Z}_T$. The resulting expression can be computed by solving the backward Kolmogorov equation, which in this case takes the form of a partial integro-differential equation (one dimension for the time and one for the space variable). This approach was developed in a fairly general setting by Vecer and Xu [110], and it was implemented by Bayraktar and Xing [12] for jump-diffusion processes.

In the current chapter we calculate $C(A_0, K, T)$ via a semi-explicit formula for the case where X is a meromorphic process. This addresses the "infinite activity problem" since we make no other restrictions on X. In particular, X may be an infinite activity, infinite variation process with double-sided jumps. We derive our formula via the technique outlined

in Section 4.2.4 involving the Mellin transform of the exponential functional. Our formula is semi-explicit in the sense that it represents a Mellin-Laplace transform of the option price. We choose this approach partly because the Mellin transform method was first pioneered for hyper-exponential processes in [27]. The relationship between hyper-exponential processes and meromorphic processes suggests that extending the technique to meromorphic processes is the most natural course of action, as opposed to attempting numerical solutions of integro-differential equations. Of course, we might also resort to Monte Carlo techniques, but these are typically computationally expensive and of limited accuracy. Our algorithm to recover the price via numerical Mellin-Laplace inversion is fast and accurate which seems to justify our choice.

The main hurdle to our approach is deriving an expression for the Mellin transform of the exponential functional of a meromorphic process. In doing so, we find that we are also able to identify the distribution of I_q. As we know from our discussion on the state of research on exponential functionals, this ancillary theoretical result is significant: it represents the first case in which the distribution of I_q is known for a process with infinite activity, infinite variation, and two-sided jumps.

To derive an expression for the Mellin transform we generalize the approach via the verification result for hyper-exponential processes of Section 4.2.3. However, for the meromorphic case, key finite quantities become infinite, and so we have to take a number of preparatory steps. In section 6.2 we study infinite products of independent beta random variables. We use these results in section 6.3 to identify the distribution of the exponential functional. This leads directly to an ancillary theoretical result: we are able to identify the distribution of the homogeneous functional of the process $L^{(1)}$ which we introduced in Example 3 of Section 4.3.3. In section 6.4 we discuss numerical issues, such as approximating the Mellin transform (which is an infinite product), and we compute option prices using our algorithm.

Throughout the chapter, as we have done previously, we will use the notation

$$\mathcal{M}(\xi, z) := \mathbb{E}[\xi^{z-1}]$$

to denote the Mellin transform of a random variable ξ.

6.2 An infinite product of beta random variables

In this section we will study infinite products of independent beta random variables. These results will be used in Section 6.3 in order to describe the distribution of the exponential functional I_q.

As before, for $a, b > 0$, let $B_{(a,b)}$ denote the beta random variable, having distribution

$$\mathbb{P}(B_{(a,b)} \in \mathrm{d}x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathrm{d}x, \quad 0 < x < 1.$$

With any two unbounded sequences $\alpha = \{\alpha_n\}_{n\geq 1}$ and $\beta = \{\beta_n\}_{n\geq 1}$ which satisfy the interlacing property

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 \ \ldots \tag{6.2}$$

we associate an infinite product of independent beta random variables, defined as

$$J(\alpha, \beta) := \prod_{n\geq 1} B_{(\alpha_n,\, \beta_n - \alpha_n)} \frac{\beta_n}{\alpha_n}. \tag{6.3}$$

The random variable $J(\alpha, \beta)$ is the main object of interest in this section. When the sequences α and β are clear from the context, we will suppress them in the notation $J(\alpha, \beta)$ and will write simply J. Our first task is to establish that this random variable is well-defined.

Proposition 1. *The infinite product in* (6.3) *converges a.s.*

Before proving Proposition 1, let us establish the following simple (but useful) result.

Lemma 3. *Assume that α and β are two unbounded sequences satisfying* (6.2)*, and $f : \mathbb{R}^+ \mapsto \mathbb{R}$ is a monotone function such that $\lim_{x\to+\infty} f(x) = 0$. Then*

$$\left| \sum_{n\geq 1} (f(\beta_n) - f(\alpha_n)) \right| < |f(\alpha_1)|\,. \tag{6.4}$$

Proof. Assume that $f(x)$ is increasing. Then the condition $f(+\infty) = 0$ implies that $f(x) \leq 0$ for all x, thus for any $m \in \mathbb{N}$ we have

$$\begin{aligned} 0 \leq S_m := \sum_{n=1}^{m} (f(\beta_n) - f(\alpha_n)) &\leq \sum_{n=1}^{m} (f(\alpha_{n+1}) - f(\alpha_n)) \\ &= f(\alpha_{m+1}) - f(\alpha_1) < -f(\alpha_1) = |f(\alpha_1)|. \end{aligned}$$

The above inequality and the fact that the sequence S_m is increasing show that the series in (6.4) converges and its sum is bounded by $|f(\alpha_1)|$. The case when $f(x)$ is decreasing can be proven in the same manner. □

Proof of Proposition 1

Taking the logarithm of both sides of (6.3), we see that we need to establish the a.s. convergence of the infinite series

$$\log(J) = \sum_{n\geq 1} \log\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right). \tag{6.5}$$

Before proceeding, we recall that the Mellin transform of a beta random variable is given by

$$\mathbb{E}[(B_{(a,b)})^{z-1}] = \frac{\Gamma(a+b)\Gamma(a+z-1)}{\Gamma(a)\Gamma(a+b+z-1)}, \quad \mathrm{Re}(z) > 1-a. \tag{6.6}$$

By differentiating the above identity twice and setting $z=1$, we find

$$\mathbb{E}[\log(B_{(a,b)})] = \psi(a) - \psi(a+b), \quad \text{and} \quad \mathrm{Var}[\log(B_{(a,b)})] = \psi'(a) - \psi'(a+b),$$

where $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function. It is known that $f(z) := \ln(z) - \psi(z)$ is a completely monotone function which decreases to zero (see Theorem 1 in [6] or Formula 8.361.8 in [49]). This implies that the function $\psi'(z) := -f'(z) + 1/z$, has the same property. Applying Lemma 3 we conclude that both the series

$$\sum_{n\geq 1}\mathbb{E}\left[\log\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right)\right] = \sum_{n\geq 1}(f(\alpha_n) - f(\beta_n)), \text{ and}$$
$$\sum_{n\geq 1}\mathrm{Var}\left(\log\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right)\right) = \sum_{n\geq 1}(\psi'(\alpha_n) - \psi'(\beta_n))$$

converge, therefore the Khintchine-Kolmogorov Convergence Theorem implies a.s. convergence of the infinite series (6.5). □

Next, we show that the Mellin transform of J exists, and has the form that we expect. To get the result, we have to work a little bit harder than we would for a finite product of independent variables.

Theorem 32. *The Mellin transform $\mathcal{M}(J,z)$ exists for all $z > 1-\alpha_1$ and it can be analytically continued to a meromorphic function*

$$\mathcal{M}(J,z) = \prod_{n\geq 1} \frac{\Gamma(\beta_n)\Gamma(\alpha_n+z-1)}{\Gamma(\alpha_n)\Gamma(\beta_n+z-1)} \left(\frac{\beta_n}{\alpha_n}\right)^{z-1}. \tag{6.7}$$

The above infinite product converges uniformly on compact subsets of the complex plane which do not contain the poles of $\mathcal{M}(J,z)$.

Proof. Proposition 1 combined with (6.6) and Lévy's Continuity Theorem imply that (6.7) is true for all z on the line $\mathrm{Re}(z) = 1$, and that the infinite product in the right-hand side of (6.7) converges uniformly on compact subsets of this line. Our first goal is to prove uniform convergence of this product on any compact subset of $\mathbb{C}$, which excludes the poles of $\mathcal{M}(J,z)$.

For $t > 0$ and $a, z \in \mathbb{C}$ satisfying $\mathrm{Re}(z) > \max(0, -\mathrm{Re}(a))$, let us define

$$f(z;a) := \log\left(\frac{\Gamma(a+z)}{\Gamma(z)z^a}\right), \quad \text{and} \quad g(t;a) := \left(a - \frac{1-e^{-at}}{1-e^{-t}}\right)\frac{1}{t}. \tag{6.8}$$

It is known that for $a \in \mathbb{R}$, the function $z \in (\max(0,-a),\infty) \mapsto |f(z;a)|$ is completely monotone. This follows from Theorem 4 in [98] if $a > 0$ and from Theorem 1 in [97] if $a < 0$. We also have the following integral representation

$$f(z;a) = \int_{\mathbb{R}^+} g(t;a)e^{-zt}\mathrm{d}t, \quad a, z \in \mathbb{C},\ \mathrm{Re}(z) > \max(0,-\mathrm{Re}(a)), \tag{6.9}$$

which can be established using Formula 8.361.5 in [49].

Let A be a compact subset of $\mathbb{C}$. Define

$$v^- := \min\{\mathrm{Re}(z) :\ z \in A\}, \quad \text{and} \quad v^+ := \max\{\mathrm{Re}(z) :\ z \in A\}.$$

The convergence of the infinite product (6.7) is not affected by any finite number of terms. Therefore, without loss of generality, we can assume that $1-\alpha_1 < v^-$. If this is not true, we may take N large enough, so that $1-\alpha_N < v^-$, and investigate the convergence of the tail $\prod_{n\geq N}(\dots)$ of the infinite product in (6.7).

For $k \geq 1$ we define

$$\mathcal{B}_k(z) := \prod_{n=1}^{k} \frac{\Gamma(\beta_n)\Gamma(\alpha_n + z - 1)}{\Gamma(\alpha_n)\Gamma(\beta_n + z - 1)} \left(\frac{\beta_n}{\alpha_n}\right)^{z-1}.$$

The function $\mathcal{B}_k(z)$ is analytic and zero-free in the half-plane $\mathrm{Re}(z) > 1 - \alpha_1$, and due to our assumption $1 - \alpha_1 < v^-$, this half-plane includes the set A. Using (6.9) and Lemma 3 we find that for all $l > k \geq 1$ and $z \in A$,

$$\begin{aligned}
|\log(\mathcal{B}_l(J, z)) - \log(\mathcal{B}_k(J, z))| &= \left| \sum_{n=k+1}^{l} \left(f(\alpha_n; z-1) - f(\beta_n; z-1)\right) \right| \\
&= \left| \int_{\mathbb{R}^+} g(t; z-1) \sum_{n=k+1}^{l} (e^{-\alpha_n t} - e^{-\beta_n t}) \mathrm{d}t \right| \\
&\leq \int_{\mathbb{R}^+} |g(t; z-1)| \sum_{n=k+1}^{\infty} (e^{-\alpha_n t} - e^{-\beta_n t}) \mathrm{d}t \\
&< \int_{\mathbb{R}^+} |g(t; z-1)|\, e^{-\alpha_{k+1} t} \mathrm{d}t.
\end{aligned}$$

For any fixed $z \in A$, the right-hand side in the above inequality can be made arbitrarily small if k is sufficiently large, since $\alpha_k \to +\infty$ as $k \to +\infty$. This shows that for $z \in A$, the collection $\{\mathcal{B}_k(z)\}_{k \geq 1}$ forms a Cauchy sequence and, therefore, the infinite product in (6.7) converges pointwise.

Now we need to establish the uniform convergence on the compact set $A \subset \mathbb{C}$. Using (6.6), we check that for all z in the half-plane $\mathrm{Re}(z) > 1 - \alpha_1$ we have $\mathcal{B}_k(z) \equiv \mathcal{M}(J_k, z)$, where J_k is defined by

$$J_k := \prod_{n=1}^{k} B_{(\alpha_n, \beta_n - \alpha_n)} \frac{\beta_n}{\alpha_n}.$$

Denoting $v = \mathrm{Re}(z)$ and using Lemma (3) and monotonicity of the function $z \in (\max(0, -a), \infty) \mapsto f(z; a)$ for $a \in \mathbb{R}$ we obtain

$$\begin{aligned}
|\mathcal{M}(J_k, z)| = |\mathbb{E}[J_k^{z-1}]| \leq \mathbb{E}[|J_k^{z-1}|] = \mathbb{E}[J_k^{v-1}] = \mathcal{M}(J_k, v) \\
= \exp\left(\sum_{n=1}^{k} (f(\alpha_n; v-1) - f(\beta_n; v-1)) \right) \\
< \exp(|f(\alpha_1; v-1)|).
\end{aligned} \tag{6.10}$$

The function $v \mapsto f(\alpha_1; v-1)$ is continuous on the interval $v \in [v^-, v^+]$, therefore it is bounded on this interval. This fact combined with the inequality (6.10) implies the uniform boundedness of all functions $\mathcal{M}(J_k, z)$, $k \geq 1$ in the vertical strip $v^- \leq \mathrm{Re}(z) \leq v^+$, therefore also on the set A. By the Vitaly-Porter Theorem, uniform boundedness and pointwise convergence of $\mathcal{M}(J_k, z)$ imply uniform convergence of these functions. This proves that the product (6.7) converges uniformly to a function which is analytic in $\mathrm{Re}(z) > 1 - \alpha_1$. By analytic continuation we conclude that the Mellin transform of $J(\alpha, \beta)$ exists everywhere in this half-plane. □

In the next proposition we show that $\mathcal{M}(J, z)$ satisfies two important identities, which will be crucial for our results on exponential functionals of meromorphic processes in Section 6.3. The approach in this upcoming section will be to show that I_q is equal in distribution to the quotient of two random variables of the form $J(\alpha, \beta)$. We will prove this using the verification result of Section 4.2.2, which, we recall, requires us to find a candidate function satisfying a functional equation involving the Laplace exponent. Our candidate function $f(z)$ will be a product of two Mellin transforms of the form $\mathcal{M}(J, z)$; the following proposition will help us establish that $f(z)$, so defined, satisfies the criteria of the verification result.

Proposition 2. *Assume that sequences α and β satisfy the interlacing property (6.2).*

(i) For $n \geq 1$ define $\tilde{\alpha}_n = \beta_n$ and $\tilde{\beta}_n = \alpha_{n+1}$. The sequences $\tilde{\alpha}$ and $\tilde{\beta}$ also satisfy the interlacing property (6.2). We have the following identity

$$\mathcal{M}(J, z) \times \mathcal{M}(\tilde{J}, z) = \frac{\Gamma(\alpha_1 + z - 1)}{\Gamma(\alpha_1)\alpha_1^{z-1}}, \quad \mathrm{Re}(z) > 1 - \alpha_1, \tag{6.11}$$

where $J := J(\alpha, \beta)$ and $\tilde{J} := J(\tilde{\alpha}, \tilde{\beta})$.

(ii) The function $\mathcal{M}(J, z)$ satisfies $\mathcal{M}(J, z+1) = \phi(z)\mathcal{M}(J, z)$ for all $z \in \mathbb{C}$, where $\phi(z)$ is a meromorphic function defined as

$$\phi(z) := \prod_{n \geq 1} \frac{1 + \frac{z-1}{\alpha_n}}{1 + \frac{z-1}{\beta_n}}, \quad z \in \mathbb{C}. \tag{6.12}$$

Proof. The proof of (i) follows at once from (6.7). Lemma 3 implies convergence of the infinite product in (6.12), and the proof of (ii) follows easily from (6.7) and the identity $\Gamma(z+1) = z\Gamma(z)$. □

6.3 Exponential functional of meromorphic processes

Let us review some important properties of meromorphic processes, which we will need for this section (see Section 2.2.3 for full details). First, the density of the Lévy measure of a meromorphic process is defined by (2.14) from which it is clear that the Lévy measure has exponentially decreasing tails.

Second, we recall that the Laplace exponent of a meromorphic process is given by

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z^2 \sum_{n\geq 1} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n\geq 1} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}, \tag{6.13}$$

where $-\hat{\rho}_1 < \mathrm{Re}(z) < \rho_1$, $\sigma \geq 0$ and $a \in \mathbb{R}$. The function $\psi(z)$ can be analytically continued to a real meromorphic function, having only simple poles at points $\{\rho_n\}_{n\geq 1}$, and $\{-\hat{\rho}_n\}_{n\geq 1}$. As we have already seen, the most important analytical property of meromorphic processes (and also hyper-exponential processes) is that for any $q > 0$ the equation $\psi(z) = q$ has only simple real solutions $\{\zeta_n\}_{n\geq 1}$, and $\{-\hat{\zeta}_n\}_{n\geq 1}$, which satisfy the interlacing property

$$\ldots - \hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \ldots. \tag{6.14}$$

As before, when we need to emphasize the dependence of these solutions on the parameter q, we will denote them by $\{\zeta_n(q)\}_{n\geq 1}$ and $\{-\hat{\zeta}_n(q)\}_{n\geq 1}$. Moreover, from Theorem 10 we have the following infinite product representation

$$q - \psi(z) = q \prod_{n\geq 1} \frac{1 - \frac{z}{\zeta_n}}{1 - \frac{z}{\rho_n}} \times \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\zeta}_n}}{1 + \frac{z}{\hat{\rho}_n}}, \quad z \in \mathbb{C}, \tag{6.15}$$

which we recognize as an expression of the Wiener-Hopf factorization; the infinite products on the right-hand side represent $(\varphi_q^+(z))^{-1} \times (\varphi_q^-(z))^{-1}$.

The next theorem is the main theoretical result in this chapter. We identify the distribution of the exponential functional I_q of a meromorphic process. The reader may wish to compare this with Theorem 14 which gives the distribution for hyper-exponential processes.

Theorem 33. *Assume that $q > 0$ and I_q is the exponential functional with respect to a meromorphic process X which has Laplace exponent $\psi(z)$. Define $\hat{\rho}_0 := 0$ and the four sequences*

$$\zeta := \{\zeta_n\}_{n\geq 1}, \quad \rho := \{\rho_n\}_{n\geq 1}, \quad \tilde{\zeta} := \{1 + \hat{\zeta}_n\}_{n\geq 1}, \quad \text{and} \quad \tilde{\rho} := \{1 + \hat{\rho}_{n-1}\}_{n\geq 1}.$$

Then we have the following identity in distribution

$$I_q \stackrel{d}{=} C(q) \times \frac{J(\tilde{\rho}, \tilde{\zeta})}{J(\zeta, \rho)}, \tag{6.16}$$

where

$$C(q) := q^{-1} \prod_{n \geq 1} \frac{1 + \frac{1}{\hat{\rho}_n}}{1 + \frac{1}{\hat{\zeta}_n}}, \tag{6.17}$$

and the random variables $J(\tilde{\rho}, \tilde{\zeta})$ and $J(\zeta, \rho)$ are independent and are defined by (6.3). $\mathcal{M}(I_q, z)$ is finite for $0 < \mathrm{Re}(s) < 1 + \zeta_1$ and is given by

$$\begin{aligned} \mathcal{M}(I_q, z) =& C(q)^{z-1} \\ & \times \prod_{n \geq 1} \frac{\Gamma(\hat{\zeta}_n + 1)\Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\rho}_{n-1} + 1)\Gamma(\hat{\zeta}_n + z)} \left(\frac{\hat{\zeta}_n + 1}{\hat{\rho}_{n-1} + 1} \right)^{z-1} \frac{\Gamma(\rho_n)\Gamma(\zeta_n + 1 - z)}{\Gamma(\zeta_n)\Gamma(\rho_n + 1 - z)} \left(\frac{\zeta_n}{\rho_n} \right)^{z-1}. \end{aligned} \tag{6.18}$$

Proof. First, we note that the two pairs of sequences (ζ, ρ) and $(\tilde{\rho}, \tilde{\zeta})$ satisfy the interlacing property (6.2), therefore the random variables $J(\zeta, \rho)$ and $J(\tilde{\rho}, \tilde{\zeta})$ are well defined. The infinite product defining constant $C(q)$ converges due to Lemma 3.

Let $f(z)$ denote the function in the right-hand side of (6.18). As in Proposition 2 we will write $\tilde{J}$ for $J(\tilde{\rho}, \tilde{\zeta})$ in order to simplify notation. Using Theorem 32 we find that $f(z) = C(q)^{z-1}\mathcal{M}(J, z)\mathcal{M}(\tilde{J}, 2 - z)$ is the Mellin transform of the random variable in the right-hand side of (6.16), and that $f(z)$ is analytic in the strip $0 < \mathrm{Re}(z) < 1 + \zeta_1$. Formula (6.18) implies that $f(z)$ is zero-free in the wider strip $-\hat{\zeta}_1 < \mathrm{Re}(z) < 1 + \rho_1$.

Our intention is to prove that $\mathcal{M}(I_q, z) \equiv f(z)$ – which implies (6.16) – via the verification result of Theorem 16. In order to complete the proof, we therefore need to show that the following three conditions are satisfied:

(i) for some $\theta > 0$, the function $f(z)$ is analytic and zero free in the vertical strip $0 < \mathrm{Re}(z) < 1 + \theta$;

(ii) the function $f(z)$ satisfies

$$f(z + 1) = \frac{z}{q - \psi(z)} f(z), \quad 0 < z < \theta; \text{ and}$$

(iii) $|f(z)|^{-1} = o(\exp(2\pi|\mathrm{Im}(z)|))$ as $\mathrm{Im}(z) \to \infty$, uniformly in the vertical strip

$$0 < \mathrm{Re}(z) < 1 + \theta.$$

According to the above discussion condition (i) is satisfied with $\theta = \zeta_1$. Let us verify condition (ii). Proposition 2 (ii) gives us two identities

$$\mathcal{M}(J, z+1) = \mathcal{M}(J, z) \times \prod_{n\geq 1} \frac{1 + \frac{z-1}{1+\hat{\rho}_{n-1}}}{1 + \frac{z-1}{1+\hat{\zeta}_n}}, \text{ and}$$
$$\mathcal{M}(\tilde{J}, 1-z) = \mathcal{M}(\tilde{J}, 2-z) \times \prod_{n\geq 1} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n}},$$

for $z \in \mathbb{C}$. Combining (6.18) with the above identities we obtain

$$\begin{aligned} f(z+1) &= C(q)^z \mathcal{M}(J, z+1)\mathcal{M}(\tilde{J}, 1-z) \\ &= f(z) \times q^{-1} \prod_{n\geq 1} \frac{1 + \frac{1}{\hat{\rho}_n}}{1 + \frac{1}{\hat{\zeta}_n}} \times \frac{1 + \frac{z-1}{1+\hat{\rho}_{n-1}}}{1 + \frac{z-1}{1+\hat{\zeta}_n}} \times \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n}}, \end{aligned}$$

where we have also used (6.17). Rearranging the terms in the above infinite product and using (6.15) we conclude that

$$f(z+1) = f(z) \times zq^{-1} \prod_{n\geq 1} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}} \times \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n}} = \frac{z}{q - \psi(z)} f(z),$$

and thus condition (ii) is also satisfied.

Finally, let us show that condition (iii) holds. We use Proposition 2 (i) and find that

$$\begin{aligned} f(z)^{-1} =& C(q)^{1-z} \times \frac{\Gamma(\zeta_1)\zeta_1^{1-z}}{\Gamma(s)\Gamma(\zeta_1 + 1 - z)} \times \\ & \mathcal{M}(J(\alpha, \beta), z) \times \mathcal{M}(J(\tilde{\alpha}, \tilde{\beta}), 2-z), \end{aligned} \tag{6.19}$$

for $0 < \mathrm{Re}(z) < 1 + \zeta_1$, where the sequences α, β, $\tilde{\alpha}$, $\tilde{\beta}$ are defined as follows

$$\alpha_n := 1 + \hat{\zeta}_n, \quad \beta_n := 1 + \hat{\rho}_n, \quad \tilde{\alpha}_n := \rho_n, \quad \text{and} \quad \tilde{\beta}_n := \zeta_{n+1}, \quad n \geq 1.$$

According to Theorem 32, the Mellin transform $\mathcal{M}(J(\alpha, \beta), z)$ (respectively $\mathcal{M}(J(\tilde{\alpha}, \tilde{\beta}), z)$) is finite in the half-plane $\mathrm{Re}(z) > -\hat{\zeta}_1$ (resp. $\mathrm{Re}(z) > 1 - \rho_1$). Since $|\mathcal{M}(J, z)| < \mathcal{M}(J, \mathrm{Re}(z))$, for any J we see that for any $\epsilon > 0$, the function $\mathcal{M}(J(\alpha, \beta), z)$ (resp. $\mathcal{M}(J(\tilde{\alpha}, \tilde{\beta}), 2-z)$) is uniformly bounded in the half-plane $\mathrm{Re}(z) \geq \epsilon - \hat{\zeta}_1$ (resp. $\mathrm{Re}(z) \leq 1 + \rho_1 - \epsilon$). Taking $\epsilon = \frac{1}{2}\min(\hat{\zeta}_1, \rho_1 - \zeta_1)$ we conclude that the function $\mathcal{M}(J(\alpha, \beta), z) \times \mathcal{M}(J(\tilde{\alpha}, \tilde{\beta}), 2-z)$ is

uniformly bounded in the vertical strip $0 \leq \mathrm{Re}(z) \leq 1 + \zeta_1$.

To estimate the gamma functions in (6.19), we use the now familiar formula

$$\lim_{y \to \infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2} - x} = \sqrt{2\pi}, \quad x, y \in \mathbb{R},$$

and recall that the limit exists uniformly in x on compact subsets of $\mathbb{R}$. The above formula shows that for any $\epsilon > 0$

$$\frac{1}{|\Gamma(z)\Gamma(\zeta_1 + 1 - z)|} = o(\exp((\pi + \epsilon)|\mathrm{Im}(z)|))$$

as $\mathrm{Im}(z) \to \infty$, uniformly in the strip $0 \leq \mathrm{Re}(z) \leq 1 + \zeta_1$. This fact combined with (6.19) and uniform boundedness of $\mathcal{M}(J(\alpha, \beta), z) \times \mathcal{M}(J(\tilde{\alpha}, \tilde{\beta}), 2 - z)$ shows that condition (iii) is also satisfied. Therefore, we have $\mathcal{M}(I_q, z) \equiv f(z)$, and this ends the proof of Theorem 33. □

We observe that the results of Theorem 33 may be useful for studying positive self-similar Markov processes (pssMps). In Example 2 of Chapter 4 we showed that we can derive an expression for the density of the supremum of a stable process via the density of exponential functional of a hypergeometric process (recall this is a particular type of meromorphic process). This approach works because of the connection between Lévy processes and pssMps via the Lamperti transform. In the example, applying the Lamperti transform to a hypergeometric process yields a stable process started at a point $x > 0$ which is killed upon exiting the upper half-plane. Similarly, the last several years have witnessed a large volume of research on self-similar Markov processes, which are constructed from stable Lévy processes by conditioning on various path transformations (see [23, 24, 71, 72, 78]). We note that in all of these examples (at least in dimension one), the Lamperti transformed process is a particular meromorphic process. Therefore, we hope that the result of Theorem 33 is useful in studying other interesting self-similar Markov processes.

As a particular example, we demonstrate that Theorem 33 allows us to identify the distribution of the homogeneous functional of the process $L^{(1)}$ which we introduced in Example 3 of Section 4.3.3. We recall that $L^{(1)}$ has initial position one, is killed upon leaving the positive half-plane, and otherwise behaves like a stable process with parameters (α, ρ). In

Section 4.3.3 we showed the following identity in distribution

$$\mathcal{A}_r \stackrel{d}{=} \int_0^{\mathbf{e}(q)} e^{(\alpha+r)\Xi_s} \mathrm{d}s,$$

where $\mathcal{A}_r$ is the homogeneous functional of $L^{(1)}$, $q = \Gamma(\alpha)/(\Gamma(\alpha(1-\rho))\Gamma(1-\alpha(1-\rho)) \geq 0$ and $(\alpha+r)\Xi$ is the meromorphic (hypergeometric) process with Laplace exponent

$$\psi(z) = -\frac{\Gamma(\alpha - z(\alpha+r))\Gamma(1+z(\alpha+r))}{\Gamma(\alpha(1-\rho) - z(\alpha+r))\Gamma(1-\alpha(1-\rho)+z(\alpha+r))} + q.$$

Applying Theorem 33 with

$$\rho_n = \frac{\alpha+n-1}{\alpha+r}, \quad \zeta_n = \frac{\alpha(1-\rho)+n-1}{\alpha+r},$$
$$\hat{\rho}_n = \frac{n}{\alpha+r}, \quad \text{and} \quad \hat{\zeta}_n = \frac{n-\alpha(1-\rho)}{\alpha+r},$$

for all $n \geq 1$ we easily obtain the following corollary. This corollary is originally due to Letemplier and Simon [81] who obtain the result by other means.

Corollary 1 (First theorem in [81])**.** *Let $\mathcal{A}_r$ be the homogeneous functional of $L^{(1)}$. Then*

$$\mathcal{A}_r \stackrel{d}{=} C \times \prod_{n\geq 0} \frac{B_{\left(1+\frac{n}{\alpha+r}, \frac{1-\alpha(1-\rho)}{\alpha+r}\right)}}{B_{\left(\frac{\alpha(1-\rho)+n}{\alpha+r}, \frac{\alpha\rho}{\alpha+r}\right)}} \times \frac{(1+r+\alpha\rho+n)(\alpha(1-\rho)+n)}{(\alpha+r+n)(\alpha+n)},$$

where

$$C = \frac{\Gamma(1+r+\alpha\rho)\Gamma(\alpha(1-\rho))}{\Gamma(1+\alpha+r)\Gamma(\alpha)},$$

Proof. Apply Theorem 33 for the process $(\alpha+r)\Xi$. To obtain C we use Euler's infinite product representation of the gamma function, i.e.

$$C = q^{-1} \prod_{n\geq 1} \frac{1+\frac{1}{\hat{\rho}_n}}{1+\frac{1}{\hat{\zeta}_n}} = \frac{\Gamma(\alpha(1-\rho))\Gamma(1-\alpha(1-\rho))}{\Gamma(\alpha)} \times \prod_{n\geq 1} \frac{1+\frac{\alpha+r}{n}}{1+\frac{\alpha+r}{n-\alpha(1-\rho)}}$$
$$= \frac{\Gamma(1+r+\alpha\rho)\Gamma(\alpha(1-\rho))}{\Gamma(1+\alpha+r)\Gamma(\alpha)}.$$

□

Remark 1

The results of Theorem 33 can be easily extended to the boundary case $q = 0$. We know the exponential functional $I_\infty = \int_0^\infty \exp(X_t)\mathrm{d}t$ is well-defined if $\mathbb{E}[X_1] < 0$. One can readily verify that the condition $\psi'(0) = \mathbb{E}[X_1] < 0$ implies $\hat{\zeta}_1(q) \to \zeta_1(0) = 0$ as $q \to 0^+$, moreover $q/\hat{\zeta}_1(q) \to |\mathbb{E}[X_1]|$. Thus the constant $C(q)$ defined by (6.17) converges as $q \to 0^+$ to

$$C(0) = \frac{1}{|\mathbb{E}[X_1]|} \prod_{n\geq 1} \frac{1 + \frac{1}{\hat{\rho}_n}}{1 + \frac{1}{\hat{\zeta}_{n+1}(0)}}.$$

Note that $\hat{\zeta}_{n+1}(0) > \hat{\rho}_n > 0$ for $n \geq 1$, therefore the above product is well defined, and it converges due to Lemma 3. The random variables $J(\tilde{\rho}, \tilde{\zeta})$ and $J(\zeta, \rho)$ are also well-defined in the limit $q \to 0^+$, provided that we identify $B_{(1,0)} \overset{d}{=} 1$.

Remark 2

The fact that I_q is the product of two independent random variables is actually not so surprising. In fact, it has been demonstrated (see [92, 93]) that for many Lévy processes the exponential functional has the same distribution as a product of two independent exponential functionals. One is the exponential functional of the dual of a subordinator and the other is the exponential functional of a spectrally positive process. Both processes are related to the Wiener-Hopf factors of the original Lévy process. Additionally, Patie and Savov [94] recently obtained some strong and general results on the Mellin transform of the exponential functional. In particular, they showed that the Mellin transform can be obtained as a generalized Weierstrass product in terms of the Wiener-Hopf factors of the underlying process. Since the Wiener-Hopf factors of meromorphic processes are infinite products of linear factors (see (6.15)), these results could lead to an alternative proof of Theorem 33. First, the Mellin transform of I_q could be expressed as a double infinite product. Then, interchanging the order in this product and using Weierstrass' infinite product representation for the gamma function could potentially provide an alternative method of deriving (6.18).

6.4 Numerical examples

For our numerical examples we will consider the θ-process, defined by the Laplace exponent

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \gamma + (-1)^j \left(c_1 \pi \left(\sqrt{(\alpha_1 + z)/\beta_1} \right)^{2j-1} \coth\left(\pi \sqrt{(\alpha_1 + z)/\beta_1} \right) \right. \tag{6.20}$$
$$\left. + c_2 \pi \left(\sqrt{(\alpha_2 - z)/\beta_2} \right)^{2j-1} \coth\left(\pi \sqrt{(\alpha_2 - z)/\beta_2} \right) \right),$$

where $j \in \{1, 2\}$, and the parameter γ is always chosen so that $\psi(0) = 0$. We will work with the two parameter sets

$$\begin{aligned} &\text{Parameter set 1: } j = 1,\ \sigma = 0.1,\ \text{and} \\ &\text{Parameter set 2: } j = 2,\ \sigma = 0.0, \end{aligned} \tag{6.21}$$

where the remaining parameters have the following common values:

$$\mu = 0.1, \quad c_1 = 0.15, \quad c_2 = 0.3, \quad \alpha_1 = \alpha_2 = 1.5, \quad \text{and} \quad \beta_1 = \beta_2 = 2.$$

Parameter set 1 defines a process with a non-zero Gaussian component and jumps of infinite activity but finite variation, while Parameter set 2 defines a process with zero Gaussian component and jumps of infinite variation (see Section 2.2.3).

6.4.1 Approximating the Mellin transform of the exponential functional

First, we discuss the problem of computing the Mellin transform and the density of the exponential functional. The algorithm we derive for computing the Mellin transform will also be useful in Section 6.4.2 where we develop a method of pricing Asian options. We start by expressing the density $p(x)$ as the inverse Mellin transform of $\mathcal{M}(I_q, z)$, that is

$$p(x) = \frac{x^{-c}}{2\pi} \int_{\mathbb{R}} \mathcal{M}(I_q, c + iu) e^{-iu \log(x)} \mathrm{d}u, \tag{6.22}$$

where c can be any number in the interval $(0, 1 + \zeta_1)$ and we have assumed that $q > 0$. We see that there are two main issues in computing $p(x)$. The first is the oscillatory nature of the integrand in (6.22), but this is easily overcome using Filon's method (see discussion in Section 5.2.2). The second issue is that we need to compute $\mathcal{M}(I_q, z)$, which given by the infinite product (6.18), to a reasonably high degree of precision. Below we present a simple

algorithm for for carrying out this computation.

Using (6.3), (6.16) and (6.17) we see that $I_q \stackrel{d}{=} I_q^{(N)} \times \epsilon^{(N)}$, where

$$I_q^{(N)} := \frac{1+\hat{\rho}_N}{q} \prod_{n=1}^{N} \frac{\zeta_n \hat{\zeta}_n}{\rho_n \hat{\rho}_n} \frac{B_{(1+\hat{\rho}_{n-1}, \hat{\zeta}_n - \hat{\rho}_{n-1})}}{B_{(\zeta_n, \rho_n - \zeta_n)}}, \tag{6.23}$$

$$\epsilon^{(N)} := \prod_{n \geq N+1} \frac{(1+\hat{\rho}_n)\zeta_n \hat{\zeta}_n}{(1+\hat{\rho}_{n-1})\rho_n \hat{\rho}_n} \frac{B_{(1+\hat{\rho}_{n-1}, \hat{\zeta}_n - \hat{\rho}_{n-1})}}{B_{(\zeta_n, \rho_n - \zeta_n)}},$$

and all random variables are assumed to be independent. From the convergence of the infinite product in (6.23) it follows that $\epsilon^{(N)} \to 1$ weakly as $N \to +\infty$, therefore the simplest way to approximate the distribution of I_q is to set $I_q = I_q^{(N)}$. In terms of Mellin transform, this results in approximating $\mathcal{M}(I_q, z)$ by

$$\mathcal{M}(I_q^{(N)}, z) = b_N^{z-1} \times \prod_{n=1}^{N} \frac{\Gamma(\hat{\zeta}_n + 1)\Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\rho}_{n-1} + 1)\Gamma(\hat{\zeta}_n + z)} \frac{\Gamma(\rho_n)\Gamma(\zeta_n + 1 - z)}{\Gamma(\zeta_n)\Gamma(\rho_n + 1 - z)}, \tag{6.24}$$

where

$$b_N := \frac{1+\hat{\rho}_N}{q} \prod_{n=1}^{N} \frac{\zeta_n \hat{\zeta}_n}{\rho_n \hat{\rho}_n}.$$

It is clear that $\mathcal{M}(I_q^{(N)}, z) \to \mathcal{M}(I_q, z)$ as $N \to +\infty$, however, the convergence may be slow, in which case we need to find a way to accelerate it. The above approximation is based on replacing the random variable $\epsilon^{(N)}$ by 1; in what follows we will try to improve it by replacing $\epsilon^{(N)}$ by the Mellin transform of a suitable random variable.

Our refined approximation is based on the observation that we can compute at least the first few moments $m_k := \mathbb{E}[(\epsilon^{(N)})^k]$ exactly. This is clear from the fact that the function $\mathcal{M}(\epsilon^{(N)}, z) = \mathbb{E}[(\epsilon^{(N)})^{z-1}]$ is analytic in the strip $-\hat{\rho}_N < \mathrm{Re}(z) < 1 + \zeta_{N+1}$, therefore the moments m_k are finite for all $k < 1 + \zeta_{N+1}$. Using the functional equation $\mathcal{M}(I_q, z+1) = z\mathcal{M}(I_q, z)/(q - \psi(z))$ and the fact that we have $\mathcal{M}(\epsilon^{(N)}, z) = \mathcal{M}(I_q, z)$ $/\mathcal{M}(I_q^{(N)}, z)$ we find

$$m_k = \frac{k!}{\mathcal{M}(I_q^{(N)}, k+1)} \prod_{j=1}^{k} \frac{1}{q - \psi(j)}, \tag{6.25}$$

and these quantities can be readily computed using (6.20) and (6.24) (in case $\psi(j)$ is equal to q or infinity for some j, we can still compute m_k via L'Hôspital's rule).

Our plan is to replace $\epsilon^{(N)}$ (whose distribution we cannot compute explicitly) by a simple random variable ξ, with known distribution and Mellin transform, so that the first two moments of ξ match the first two moments of $\epsilon^{(N)}$. We will take ξ to be a beta random variable of the second kind, which is defined by its density

$$\mathbb{P}(\xi \in \mathrm{d}x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} y^{a-1}(1+y)^{-a-b}\mathrm{d}y, \quad y > 0,$$

where the parameters a and b must be positive. The Mellin transform of ξ is given by

$$\mathbb{E}[\xi^{z-1}] = \frac{\Gamma(a+z-1)\Gamma(b+1-z)}{\Gamma(a)\Gamma(b)}.$$

One can check that if we define the parameters

$$a = m_1 \frac{m_1+m_2}{m_2-m_1^2}, \quad b = 1 + \frac{m_1+m_2}{m_2-m_1^2}, \tag{6.26}$$

then we have $\mathbb{E}[\xi] = m_1$ and $\mathbb{E}[\xi^2] = m_2$.

Let us summarize the proposed algorithm for approximating the Mellin transform $\mathcal{M}(I_q, z)$. We set N to be a large number (large enough so that the condition $\zeta_{N+1} > 1$ is satisfied). Then m_1 and m_2 are finite, and we compute these numbers using (6.25). We evaluate the parameters a and b via (6.26) and approximate the Mellin transform $\mathcal{M}(I_q, z)$ by

$$\mathcal{M}(I_q^{(N)}, z) \frac{\Gamma(a+z-1)\Gamma(b+1-z)}{\Gamma(a)\Gamma(b)}, \tag{6.27}$$

where $\mathcal{M}(I_q^{(N)}, z)$ is given by (6.24). We emphasize that (6.27) is the Mellin transform of a random variable $I_q^{(N)} \times \xi$, which converges to I_q in distribution as $N \to +\infty$. Provided the first two moments of I_q exist, $I_q^{(N)} \times \xi$ will have the same first two moments; if the classical moments do not exist, then we can think of moment matching taking place in terms of "analytically continued" moments.

We illustrate the efficiency of this approximation by a numerical example. We compute the density of the exponential functional $I_{\mathbf{e}(1)}$ for the two parameter sets (6.21), using ap-

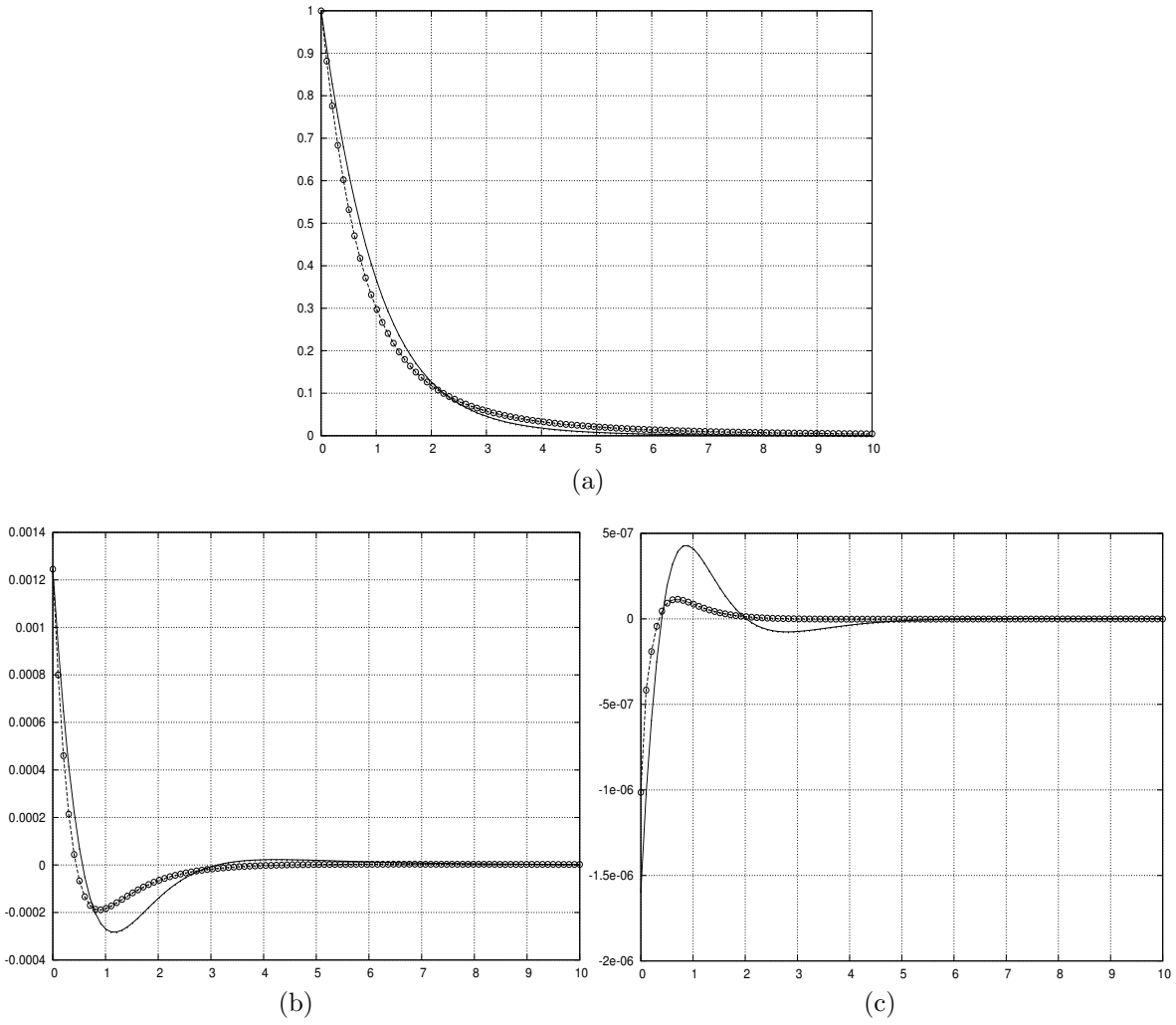


Figure 6.1: (a) The density of the exponential functional $I_{\mathbf{e}(1)}$ with $N = 400$ (the benchmark). (b) The error with $N = 20$ (no correction). (c) The error with $N = 20$ (with correction term). Solid lines (resp. circles) represent parameter set I (resp. II).

proximation (6.24) with $N = 400$. The graphs of the densities are shown in figure 6.1 (a), and we take these results as our benchmark. Then we calculate the density (using the same approximation (6.24)) with $N = 20$; the error between these results and our benchmark is shown in figure 6.1 (b). We see that the maximum error is of the order 1.0e-3. Finally, we perform the same calculation with $N = 20$, but now we use the approximation for the Mellin transform with the correction term (6.27). In figure 6.1(c) we see that the maximum error is of the order 1.0e-6; therefore our approximation (6.27) decreases the error by a factor of 1000, and seems to be very efficient.

6.4.2 Computing the price of an Asian option

Let us briefly review the pricing algorithm we outlined in Sections 4.1.1 and 4.2.4 in the context of our current example. Specifically, we will demonstrate that all key quantities are finite and well defined. In this scenario, we are working with the stock price process $A_t = A_0 e^{X_t}$ where X is a θ-process of the form (6.20). To ensure that our probability measure is risk neutral, we adjust the drift μ of our process such that the risk-neutral condition $\psi(1) = r$ holds (note we also assume $\rho_1 > 1$ otherwise we may have $\psi(1) = +\infty$).

From Section 4.1.1 we know that calculating $C(A_0, K, T)$ is equivalent to calculating the value of the function $f_a(k,t) = \mathbb{E}[(I_t - k)^+]$ where k and t are just simple transformations of K and T. Therefore, we will work with the simpler $f_a(k,t)$ and apply the Laplace-Mellin transform method of Cai and Kou [27]. First we take the Laplace transform of $f_a(k,t)$ in the t variable, or equivalently we evaluate $f_a(k,t)$ at the time $\mathbf{e}(q)$, to obtain the function

$$h_a(k,q) = q \int_{\mathbb{R}^+} e^{-qt} f_a(k,t) \mathrm{d}t = \mathbb{E}[(I_q - k)^+]. \tag{6.28}$$

In the context of the current example the function $h_a(k,q)$ will be finite for $q > r$ since $\mathbb{E}[(I_q - k)^+] < \mathbb{E}[I_q] = (q-r)^{-1}$, where the last equality follows from functional identity (ii) of the verification result, and the risk-neutral condition $\psi(1) = r$. From Section 4.2.4 and Theorem 33, we also know that for any $z \in (0, \zeta_1 - 1)$ we may derive the following expression for the Mellin transform of $h_a(k,q)$:

$$\Phi(z,q) = \frac{\mathcal{M}(I_q, z+2)}{z(z+1)}. \tag{6.29}$$

Therefore, what remains to show is that the interval $(0, \zeta_1 - 1)$ is not empty. We can demonstrate the equivalent statement, that $\zeta_1 > 1$, by noting that: a) $\psi(z)$ is convex on $(-\rho_1, \rho_1)$;

b) $\psi(0) = 0$; c) ζ_1 is the smallest solution of $\psi(z) = q$, and $\zeta_1 < \rho_1$; d) $\psi(1) = r$ according to our risk-neutrality assumption; and finally e) $h_a(k, q)$ is defined only for $q > r$. Taking these facts together (see Figure 6.2) shows $\zeta_1 > 1$ and therefore that $\Phi(z, q)$ is finite on the non-empty interval $(0, \zeta_1 - 1)$.

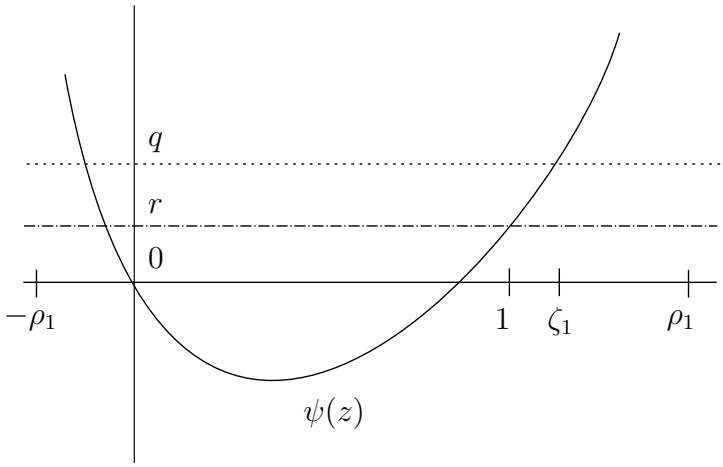


Figure 6.2: Visual demonstration that $\zeta_1 > 1$.

We now compare three different algorithms for computing the price of an Asian option when the stock price is driven by a θ-processes which satisfies the conditions we have just discussed (i.e. $\psi(1) = r$, and $\rho_1 > 1$). The first algorithm is based on inverting $\Phi(z, q)$ using our approximation of the Mellin transform (6.27). The second is based on approximating the θ-process by a hyper-exponential process and then deriving and inverting $\Phi(z, q)$ for this approximation. We recall from Chapter 4 that we have a closed form expression (4.14) for the Mellin transform of the exponential functional of a hyper-exponential process. Lastly, we calculate $C(A_0, K, T)$ directly via a Monte Carlo simulation.

Algorithm 1: Approximating the Mellin transform of the exponential functional

This algorithm is based on inverting the Laplace and Mellin transforms in (6.28) and (6.29) and approximating $\mathcal{M}(I_q, s)$ by the algorithm presented in Section 6.4.1. First, for $d_2 > r$ and $q = d_2 + iu$ we compute $h_a(k, q)$ as the inverse Mellin transform

$$h_a(k, q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(I_q, d_1 + iv + 2)}{(d_1 + iv)(d_1 + iv + 1)} e^{-iv \log(k)} \mathrm{d}v, \tag{6.30}$$

where $d_1 \in (0, \zeta_1(d_2) - 1)$. Second, we compute $f_a(k,t)$ as the inverse Laplace transform, which can be rewritten as the cosine transform

$$f_a(k,t) = \frac{2e^{d_2 t}}{\pi} \int_{\mathbb{R}^+} \mathrm{Re}\left(\frac{h_a(k, d_2 + iu)}{d_2 + iu}\right) \cos(ut)\mathrm{d}u. \tag{6.31}$$

We set $d_1 = d_2 = 0.25$ and truncate the integral in (6.30) (resp. (6.31)) so that the domain of numerical integration is $-100 < v < 100$ (resp. $0 < u < 200$), and use Filon's method (Section 5.2.2) with 400 discretization points to evaluate each of these integrals. The Mellin transform is computed using the approximation algorithm presented in section 6.4.1, with the Mellin transform truncated at N terms (we will set $N \in \{10, 20, 40, 80\}$ in our computations). Computing the Mellin transform requires computing $2N$ solutions to the equation $\psi(z) = q$. See Section 5.2.3 for a discussion of the numerical aspects of this computation.

In Section 5.2.3 we highlighted some of the practical issues (and their solutions) with computing the values of $\mathcal{M}(I_q, z)$ for complex values of q, in particular, the issues with finding solutions of the equation $\psi(z) = q$. Here we also uncover a theoretical issue with our algorithm, namely that the formula for $\mathcal{M}(I_q, z)$ from Theorem 33 is only valid for $q > 0$. To extend the results of Theorem 33, the main step would be to establish the uniform convergence of the infinite product on the right-hand side of (6.18) for complex q. For this we would need some additional information about the behavior of the solutions $\{\zeta_n\}_{n\geq 1}$, $\{-\hat{\zeta}_n\}_{n\geq 1}$ to the equation $\psi(z) = q$ when q is complex. A rigorous discussion of this question is beyond the scope of the present work; we see that our calculations support the conjecture that the formula holds also for $q \in \mathbb{C}$.

Algorithm 2: Approximation by a hyper-exponential process.

Our second algorithm is based on approximating the theta-process X by a hyper-exponential process $\tilde{X}^{(N)}$, for which the Mellin transform of the exponential functional can be computed explicitly (4.14). The procedure for approximating X by $X^{(N)}$ is simple: We truncate both infinite series defining the Lévy measure of a meromorphic process at N terms (see [35] for another approximation technique). This gives us a Lévy process $\tilde{X}^{(N)}$ with hyper-exponential jumps, whose Laplace exponent is given by

$$\tilde{\psi}(z) = \frac{\tilde{\sigma}^2 z^2}{2} + \tilde{\mu} z + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n=1}^{N} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}; \tag{6.32}$$

see (6.13) for comparison. We see that the coefficients $\{a_n,\, \rho_n\}_{1\le n\le N}$ and $\{\hat{a}_n,\, \hat{\rho}_n\}_{1\le n\le \hat{N}}$ are just those of the original meromorphic process. To determine the remaining parameters of $\tilde{\psi}(z)$ we proceed as follows: $\tilde{\sigma}$ is chosen so that the variance of $\tilde{X}_t^{(N)}$ matches the variance of X_t, which is equivalent to requiring $\tilde{\psi}''(0) = \psi''(0)$; the parameter $\tilde{\mu}$ is then specified by enforcing the risk-neutral condition $\tilde{\psi}(1) = r$.

Now we can compute the price of the Asian option, with the driving process $\tilde{X}^{(N)}$, following the same procedure as for Algorithm 1. The only difference is that the Mellin transform of the exponential functional $I_q(\tilde{X})$ can be expressed in closed form. From Section 4.2.3 we recall that

$$\mathcal{M}(I_q(\tilde{X}), z) = a \times \left(\frac{\tilde{\sigma}^2}{2}\right)^{1-z} \times \Gamma(s) \times \frac{\prod_{j=1}^{N} \Gamma(\hat{\rho}_j + z)}{\prod_{j=1}^{N+1} \Gamma(\hat{\zeta}_j + z)} \times \frac{\prod_{j=1}^{N+1} \Gamma(1 + \zeta_j - z)}{\prod_{j=1}^{N} \Gamma(1 + \rho_j - z)}, \tag{6.33}$$

where $a = a(q)$ is chosen so that $\mathcal{M}(I_q(\tilde{X}), z) = 1$, $\{\zeta_n\}_{1\le n\le N+1}$ and $\{-\hat{\zeta}_n\}_{1\le n\le \hat{N}+1}$ are the solutions of the equation $\tilde{\psi}(z) = q$, and $\{\rho_n\}_{1\le n\le N}$ and $\{-\hat{\rho}_n\}_{1\le n\le \hat{N}}$ are the poles of $\tilde{\psi}(z)$. Once we have (6.33) we can calculate the equivalent of expressions (6.30) and (6.31) numerically using Filon's method and the same domains of integration as we used for Algorithm 1.

Algorithm 3: Monte-Carlo simulation.

We will also check the accuracy of the previous two algorithms by computing the price via a simple Monte-Carlo simulation. We approximate the θ-process $X = \{X_t\}_{0\le t\le T}$ by a random walk $Z = \{Z_n\}_{0\le n\le 400}$ with $Z_0 = 0$ and the increment $Z_{n+1} - Z_n \overset{d}{=} X_{T/400}$. The price of the Asian option is then approximated by the following expectation

$$e^{-rT}\mathbb{E}\left[\left(\frac{1}{400}\sum_{n=1}^{400} A_0 e^{Z_n} - K\right)^+\right],$$

which we estimate by sampling 10^6 paths of the random walk. In order to sample from the distribution of $Y := Z_{n+1} - Z_n$, we compute its density $p_Y(x)$ via the inverse Fourier transform

$$p_Y(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}[e^{izY}]e^{-izx}\mathrm{d}z,$$

N	Alg. 1, price	Alg. 1, time (sec.)	Alg. 2, price	Alg. 2, time (sec.)
10	4.724627	1.6	4.720675	1.2
20	4.727780	2.8	4.728032	1.8
40	4.728013	4.8	4.728031	3.4
80	4.728029	9.2	4.728031	7.1

Table 6.1: The price of the Asian option, Parameter set 1. The Monte Carlo estimate of the price is 4.7386 with a standard deviation of 0.0172. The exact price is 4.72802±1.0e-5.

N	Alg. 1, price	Alg. 1, time (sec.)	Alg. 2, price	Alg. 2, time (sec.)
10	10.620243	1.6	10.621039	1.2
20	10.620049	3.0	10.620171	2.2
40	10.620037	4.8	10.620054	3.6
80	10.620036	9.6	10.620039	7.4

Table 6.2: The price of the Asian option, Parameter set 2. The Monte Carlo estimate of the price is 10.6136 with the standard deviation 0.0251. The exact price is 10.62003±1.0e-5.

where $\mathbb{E}[e^{izY}] = \mathbb{E}[e^{izX_{T/400}}] = \exp((T/400)\psi(iz))$ and the Laplace exponent $\psi(z)$ is given by (6.20). Again, in order to compute the inverse Fourier transform, we use Filon's method with 10^6 discretization points.

Discussion of the results

We compute the price of an Asian option with the initial stock price $A_0 = 100$, interest rate $r = 0.03$, maturity $T = 1$, and strike price $K = 105$. We consider the two θ-processes defined by Parameter sets 1 and 2 (see (6.21)). Note that the parameter μ is not equal to 0.1 anymore, as it is determined by the risk-neutral condition $\psi(1) = r$. The results of these computations are presented in Tables 6.1 and 6.2. The code is written in Fortran90, and all computations are performed on a basic laptop (CPU: Intel Core i5-2540M 2.60GHz).

The results presented in Tables 6.1 and 6.2 show that both Algorithm 1 and Algorithm 2 perform very well, and seem to converge quickly to the true value of the option. The exact price is computed with $N = 160$ and 1600 discretization points for the two integrals in (6.30) and (6.31) using Algorithm 1. The convergence rates are similar for other choices of parameters, leading us to conclude that these values are correct to within 1.0e-5. Both of these algorithms are very efficient; the CPU time is comparable with the results of Cai and Kou on hyper-exponential processes (see [27]). Note that there is a substantial difference between Algorithm 1 and Algorithm 2, as one is based on approximating the Mellin transform of the exponential functional, and the other is based on approximating the underlying Lévy process by a hyper-exponential process. Yet, the results of the algorithms agree up to five decimal digits, which is a good indicator that they are indeed correct. The algorithm based on the Monte Carlo simulation also produces consistent results, however these estimates are much less accurate and require CPU time on the order of several minutes.

Prices were also calculated for many other values of the parameters of the underlying θ-process, as well as for different maturities and different strike prices. Qualitatively, the results seem to be consistent with the ones presented in tables 6.1 and 6.2. Algorithm 2 is very efficient for Parameter set 1, and gives high accuracy even for relatively small values of N. This is not surprising, as in this case we have a finite variation, infinite activity θ-process. It is intuitively clear that processes with compound Poisson jumps can provide a good approximation for such processes. On the other hand, Parameter set 2 corresponds to an infinite variation θ-process with zero Gaussian component. Here, our approximation by a compound Poisson process with a non-zero Gaussian component will clearly be less precise. Nevertheless, Algorithm 2 works quite well in all cases; it may well be the best method to compute prices of Asian options for all meromorphic processes (though testing on other meromorphic processes, such as β-processes would also be worthwhile).

Algorithm 1 has comparable performance, and although it is a little slower than Algorithm 2, it has one potential advantage. Suppose we want to compute the price of an Asian option for $N = 20$ and then to check whether we have sufficient accuracy by doing the same computation with $N = 40$. Algorithm 2 would require re-computing everything, since the Laplace exponent $\tilde{\psi}(z)$ of the hyper-exponential process will change, thus all the numbers $\{\zeta_n\}_{1\le n\le N+1}$, $\{-\hat{\zeta}_n\}_{1\le n\le \hat{N}+1}$ (the solutions of $\tilde{\psi}(z) = q$) will be different. This is not the case for Algorithm 1. Here the Laplace exponent $\psi(z)$ does not depend on N, thus the 20 smallest solutions of $\psi(z) = q$ will not be affected. Therefore, we only need to compute the 20 next smallest solutions, as opposed to computing 40 new solutions. The same idea can

be applied in evaluating the Mellin transform, where the results for $N = 20$ in (6.24) can be stored in memory; only the remaining finite product of gamma functions with $21 \leq n \leq 40$ has to be evaluated.

Chapter 7

The density of the supremum of a stable process

An absolutely convergent double series representation for the density of the supremum of an α-stable Lévy process is given in Theorem 2 in [58] for almost all irrational α. That is, when $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ (see Section 3.2.2 for a definition of the set $\mathcal{L}$) the absolutely convergent double series representation holds. This result cannot be made stronger in the following sense: the series does not converge absolutely for α belonging to a dense subset of $\mathcal{L}$ (Theorem 2 in [70]). Our main result in this short chapter shows that *for every* irrational α there is a way to rearrange the terms of the double series so that it converges to the density of the supremum. We show how one can establish this stronger result by introducing a simple yet non-trivial modification in the original proof of Theorem 2 in [58].

7.1 Introduction

We begin with a quick review of the necessary details of stable processes and the current state of knowledge about the density of the running supremum process (see also Sections 2.2.1, 3.2.2, and 4.3.3). We recall that an α-stable Lévy process X is defined by the characteristic exponent

$$\Psi(z) = \mathbb{I}(z < 0)e^{-\pi i\alpha(1-2\rho)/2}|z|^{\alpha} + \mathbb{I}(z > 0)e^{\pi i\alpha(1-2\rho)/2}|z|^{\alpha}, \quad z \in \mathbb{R}. \tag{7.1}$$

For the reasons discussed in Section 2.2.1 we assume that the parameters (α, ρ) belong to the set of admissible parameters

$$\mathcal{A} = \{\alpha \in (0,1), \rho \in (0,1)\} \cup \{\alpha = 1, \rho = \tfrac{1}{2}\} \cup \{\alpha \in (1,2), \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}.$$

As usual, we denote the running supremum process by S, where $S_t := \sup\{X_s : 0 \leq s \leq t\}$. We will denote the density of S_1 by $p(x)$, and we remark that due to the self-similarity property of stable processes (2.6), we have $S_t \stackrel{d}{=} t^{\frac{1}{\alpha}} S_1$. Therefore, we lose no generality by concentrating only on $p(x)$, which is the principal object of interest in this chapter. Additionally, we recall one other consequence of the self-similarity property, namely that for the positive Wiener-Hopf factor $\varphi_q^+(z)$ we have $\varphi_q^+(z) = \varphi_1^+(zq^{-1/\alpha})$. This allows us to work with the function $\varphi(z) = \varphi_1^+(-z)$, again without loss of generality.

Lastly, we remember the definition of the set of irrational numbers $\mathcal{L}$. This set consists of numbers x which satisfy

$$\left| x - \frac{p}{q} \right| < \frac{1}{b^q},$$

for at least one $b > 1$, and infinitely many coprime integers p and q. We can show (see [58,68]) that although $\mathcal{L}$ is dense in $\mathbb{R}$ it is "small" in the sense that it has Hausdorff dimension zero, and therefore also Lebesgue measure zero. It is also a subset of the Liouville numbers. In Section 3.2.2 we saw that the form of $\varphi(z)$ depends crucially on the arithmetic properties of α, in particular, for irrational α it is important to know whether α is in $\mathcal{L}$. Next we establish the connection between $\varphi(z)$ and $p(x)$ and show that the same is true for $p(x)$.

7.1.1 The current state of research

As stated, the main object of interest in this chapter is the density $p(x)$. There are several very recent results concerning this function that follow directly from the results of Section 3.2.2 on the Wiener-Hopf factorization for stable processes. However, as we shall see, there is still an unsatisfying gap in the current state of knowledge which will be addressed in this chapter. Before reviewing the history of $p(x)$, we demonstrate the connection with the Wiener-Hopf factor $\varphi(z)$.

First, we recall the obvious fact that for any Lévy process with running supremum S we

have

$$\varphi_q^+(z) = \mathbb{E}[e^{zS_{\mathbf{e}(q)}}] = \int_{\mathbb{R}^+} qe^{-qt}\mathbb{E}[e^{zS_t}]\mathrm{d}t, \quad q > 0,\ \mathrm{Re}(z) \leq 0,$$

so that information about the distribution of S_t is theoretically available from inverting the Laplace transform. However, since the form of $\varphi_q^+(z)$ does not, in general, permit analytical inversion, we can only solve this problem numerically.

For stable processes, an alternate way forward is via the Mellin transforms of S_1 and $\varphi(z)$, which we define by

$$\mathcal{M}(S_1, w) := \mathbb{E}[S_1^{w-1}], \quad 1 - \alpha\rho < \mathrm{Re}(w) < 1 + \alpha,$$

and

$$\Phi(w) := \int_{\mathbb{R}^+} z^{w-1}\varphi(z)\mathrm{d}z, \quad 0 < \mathrm{Re}(w) < \alpha\rho$$

respectively. The link between these two functions, established in Section 6 in [68] via the self-similarity property, is that

$$\Phi(w) = \Gamma(w)\Gamma\left(1 - \frac{w}{\alpha}\right)\mathcal{M}(S_1, 1 - w), \quad 0 < \mathrm{Re}(w) < \alpha\rho. \tag{7.2}$$

Remarkably, the analytic properties that $\varphi(z)$ inherits from the double-gamma function (see Theorem 7) allow us to derive an explicit expression for $\mathcal{M}(S_1, w)$ via (7.2) for any stable process.

Theorem 34 (Theorem 8 in [68])**.** *For $w \in \mathbb{C}$,*

$$\mathcal{M}(S_1, w) = \alpha^{w-1}\frac{G(\alpha\rho;\alpha)}{G(\alpha(1-\rho)+1;\alpha)} \times \frac{G(\alpha(1-\rho)+2-w;\alpha)}{G(\alpha\rho-1+w;\alpha)} \times \frac{G(\alpha-1+s;\alpha)}{G(\alpha+1-w;\alpha)}, \tag{7.3}$$

where $G(z;\tau)$ is the double-gamma function (see Appendix A).

Omitted from the statement of Theorem 34, but included in the original work, is a simplified expression for $\mathcal{M}(S_1, w)$ for a process in one of the Doney classes $\mathcal{C}_{k,l}$. In this case (7.3) reduces to a finite product of sine and gamma functions. Most important for our purposes is that $\mathcal{M}(S_1, w)$ is quasiperiodic with periods 1 and α. In particular, from Theorem 7 in [68]

we know that it satisfies the following recursive formula

$$\mathcal{M}(S_1, w+1) = \frac{\alpha}{\pi} \sin\left(\pi\left(\rho - \frac{1-w}{\alpha}\right)\right) \Gamma\left(1 - \frac{w}{\alpha}\right) \Gamma\left(1 - \frac{1-w}{\alpha}\right) \mathcal{M}(S_1, w). \tag{7.4}$$

To derive a formula for $p(x)$ via $\mathcal{M}(S_1, w)$ we proceed as usual, that is we invert the Mellin transform to obtain the formula

$$p(x) = \frac{1}{2\pi i} \int_{1+i\mathbb{R}} \mathcal{M}(S_1, w) x^{-w} \mathrm{d}w.$$

Surprisingly, we find that we can invert $\mathcal{M}(S_1, w)$ analytically for many choices of the admissible parameters (see full details in the Summary section below). To do so we rely on the following facts: a) $\mathcal{M}(S_1, w)$ can be extended to a meromorphic function (Lemma 2 in [68]) and b) $\mathcal{M}(S_1, w)$ decays exponentially as $|\mathrm{Im}(w)| \to \infty$. These suggest using the (infinite) contours shown in Figure 7.1 and the residue theorem to derive the following expression

$$p(x) = \sum_{\substack{\lambda \in \Lambda \\ c < \mathrm{Re}(\lambda) < 1}} \mathrm{Res}(\mathcal{M}(S_1, w), \lambda) x^{-\lambda} + \frac{1}{2\pi i} \int_{c+i\mathbb{R}} \mathcal{M}(S_1, w) x^{-w} \mathrm{d}w, \tag{7.5}$$

where Λ is the set of poles of $\mathcal{M}(S_1, w)$, and $c < 1$. Of course, we have a similar expression for $c > 1$. In employing this technique the hope is that as $|c| \to \infty$ the integral on the right-hand side of (7.5) vanishes, and the sum becomes a convergent series for all $x > 0$. Ultimately, whether or not this method yields an explicit expression for $p(x)$ depends crucially on the poles and residues of $\mathcal{M}(S_1, w)$, which in turn depend on the parameter α. Below we summarize what is currently known about $p(x)$ for a general stable process, i.e. a stable process for which we make no restrictions on the direction or intensity of the jumps. All results are based on the method we have described [1]. As mentioned in Section 3.2.2, more is known for processes with just one-sided jumps, but these are not the focus of the current work.

Summary

(i) The simplest case is when X belongs to one of the Doney classes $\mathcal{C}_{k,l}$. In this case an absolutely convergent series representation for $p(x)$ is given in Theorem 10 in [68]. From now on, we assume $X \notin \mathcal{C}_{k,l}$.

[1] Recall from Example 2 of Section 4.3.3 that we can also approach the problem via the density of the exponential functional of a hypergeometric process.

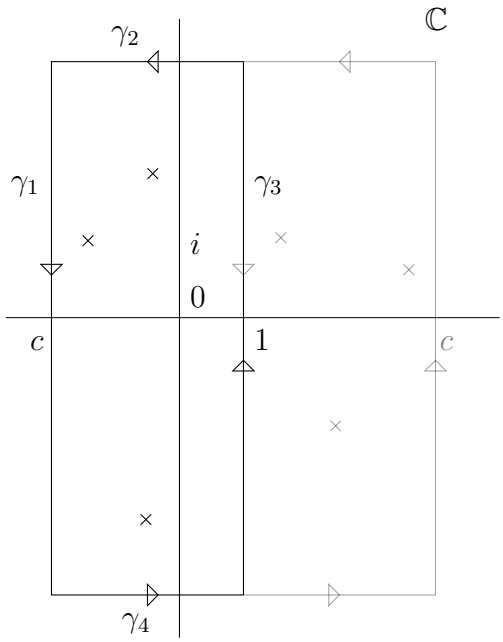


Figure 7.1: We choose a point c progressively further to the left (black) or right (grey) of the point 1. Integrating over the rectangular contour drawn in black, and letting the vertical lines γ_1 and γ_3 extend to infinity, we get (7.5). Formula (7.5) is given in terms of the residues at the poles (poles marked with x's) which have real part less than 1 and greater than c.

(ii) When α is rational, Theorem 3 in [70] gives an explicit formula for $\mathcal{M}(S_1, w)$, which is expressed in terms of elementary functions and the dilogarithm function $\mathrm{Li}_2(w)$. This expression is amenable to numerical inversion techniques. Unfortunately, $\mathcal{M}(S_1, w)$ has poles of multiplicity greater than one, and computing their residues in closed form seems to be impossible.

(iii) When α is irrational and not in $\mathcal{L}$, the function $\mathcal{M}(S_1, w)$ has simple poles at the points

$$\{s^+_{m,n}\}_{m\geq 1, n\geq 1} = \{m + \alpha n\}_{m\geq 1, n\geq 1}, \text{ and} \tag{7.6}$$

$$\{s^-_{m,n}\}_{m\geq 1, n\geq 1} = \{1 - \alpha\rho - m - \alpha n\}_{m\geq 1, n\geq 1} \tag{7.7}$$

which have residues

$$\mathrm{Res}(\mathcal{M}(S_1, w), s^+_{m,n}) = -b_{m-1,n}, \quad \text{and} \quad \mathrm{Res}(\mathcal{M}(S_1, w), s^-_{m,n}) = a_{m,n},$$

where

$$a_{m,n} := \frac{(-1)^{m+n}}{\Gamma\left(1-\rho-n-\frac{m}{\alpha}\right)\Gamma(\alpha\rho+m+\alpha n)} \times \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi}{\alpha}\left(\alpha\rho+j-1\right)\right)}{\sin\left(\frac{\pi j}{\alpha}\right)} \prod_{j=1}^{n} \frac{\sin(\pi\alpha(\rho+j-1))}{\sin(\pi\alpha j)}, \tag{7.8}$$

and

$$b_{m,n} := \frac{\Gamma\left(1-\rho-n-\frac{m}{\alpha}\right)\Gamma(\alpha\rho+m+\alpha n)}{\Gamma\left(1+n+\frac{m}{\alpha}\right)\Gamma(-m-\alpha n)} a_{m,n}. \tag{7.9}$$

The following key theorem is due to Hubalek and Kuznetsov.

Theorem 35 (Theorem 2 in [58]). *Assume that $\alpha \notin \mathbb{Q} \cup \mathcal{L}$. Then for all $x > 0$*

$$p(x) = x^{-1-\alpha} \sum_{n\geq 0} \sum_{m\geq 0} b_{m,n+1} x^{-m-\alpha n} \ , \text{ if } \alpha \in (0,1), \text{ and} \tag{7.10}$$

$$p(x) = x^{\alpha\rho-1} \sum_{n\geq 0} \sum_{m\geq 0} a_{m,n} x^{m+\alpha n} \quad , \text{ if } \alpha \in (1,2), \tag{7.11}$$

where each series converges absolutely.

(iv) Theorem 2 in [70] states that the result of Theorem 35 cannot be substantially improved: there exists an uncountable dense subset $\tilde{\mathcal{L}} \subset \mathcal{L}$, such that for all $\alpha \in \tilde{\mathcal{L}}$ and almost all ρ the series in (7.10) and (7.11) do not converge absolutely for all $x > 0$.

The current situation is clearly deficient, since we do not have a useful expression for $p(x)$ if $\alpha \in \tilde{\mathcal{L}}$ and we do not know whether the series (7.10) and (7.11) converge if $\alpha \in \mathcal{L} \setminus \tilde{\mathcal{L}}$. Also, determining whether α belongs to $\mathcal{L}$ or $\tilde{\mathcal{L}}$ is problematic, as this would require full knowledge of the continued fraction representation (see definition below) of α (see [58, Proposition 1] and [70, Proposition 1]). Since it is impossible to have absolute convergence of the series for all irrational α, the only remaining possibility is to try to find a conditionally convergent version of (7.10) and (7.11). This means finding a specific order of summation such that the series converge. Our main result shows that the right way to compute the partial sums in (7.10) and (7.11) is over triangles $\{(m,n) : m,\, n \geq 0,\, 0 \leq m + \alpha n < c_k\}$ where $\{c_k\}_{k\geq 1}$ is a known, unbounded, and increasing sequence which depends crucially on the arithmetic properties of α. When summed in this way the partial sums converge to $p(x)$ for all irrational α.

7.2 Main result

To state the main result, we must first review a (very) small amount of the theory of continued fractions. For $x \in \mathbb{R}$, let $[x]$ denote the largest integer not greater than x and let $\{x\} := x - [x]$ denote the fractional part of x. The ***continued fraction representation*** (see [62]) of x is defined as

$$x = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}},$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$. The coefficients of the continued fraction can be computed recursively as follows: define $x_1 := \{x\}$ and $x_{i+1} := \{1/x_i\}$, $i \geq 1$, then $a_0 = [x]$ and $a_i = [1/x_i]$, $i \geq 1$. For $x \notin \mathbb{Q}$ the continued fraction representation has infinitely many terms; truncating it after n steps results in a rational number $p_n/q_n := [a_0; a_1, a_2, ..., a_n]$, which provides the best rational approximation of x (among all rational numbers with denominators not greater than q_n) and is called the ***nth convergent*** (see Theorem 17 in [62]). The numerators and denominators of convergents are known to satisfy the two-term recurrence relation

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, & p_{-1} = 1, \quad p_{-2} = 0, \\ q_n = a_n q_{n-1} + q_{n-2}, & q_{-1} = 0, \quad q_{-2} = 1. \end{cases} \tag{7.12}$$

In particular, we see that $\{q_n\}_{n \geq 1}$ forms an increasing, unbounded sequence with $q_n \geq n$. With the important concepts in place, we can state our main result.

Theorem 36. *Assume that $\alpha \notin \mathbb{Q}$. Then for all $x > 0$*

$$p(x) = \begin{cases} x^{-1-\alpha} \lim\limits_{k \to \infty} \sum\limits_{\substack{m+1+\alpha(n+\frac{1}{2}) < q_k \\ m \geq 0,\, n \geq 0}} b_{m,n+1} x^{-m-\alpha n}, & \text{if } \alpha \in (0,1), \\ x^{\alpha\rho-1} \lim\limits_{k \to \infty} \sum\limits_{\substack{m+1+\alpha(n+\frac{1}{2}) < q_k \\ m \geq 0,\, n \geq 0}} a_{m,n} x^{m+\alpha n}, & \text{if } \alpha \in (1,2), \end{cases} \tag{7.13}$$

where $a_{m,n}$ and $b_{m,n}$ are defined by (7.8) and (7.9) and $q_k = q_k(2/\alpha)$ is the denominator of the kth convergent for $2/\alpha$.

To obtain the proof of Theorem 36 we need to modify one step in the proof of Theorem 35. In order to provide the reader with insight into our method, we provide a sketch of the proof

of the latter theorem. We follow the original proof from [58], but pay special attention to the step that requires modification.

Sketch of the proof of Theorem 35

Let us assume that $\alpha \in (1,2)$ and that $\alpha \notin \mathbb{Q} \cup \mathcal{L}$. Setting $c = c_k = 1 - \alpha\rho + \frac{\alpha}{2} - k$ for $k \geq 1$ in (7.5), and using the expressions for the poles and residues (7.6) and (7.8) we may derive the following equality

$$p(x) = x^{\alpha\rho - 1} \sum_{\substack{m+\alpha(n+\frac{1}{2})<k \\ m\geq 0,\, n\geq 0}} a_{m,n} x^{m+\alpha n} + \mathrm{e}_k(x), \tag{7.14}$$

where,

$$\begin{aligned} \mathrm{e}_k(x) :=& \frac{1}{2\pi i} \int_{c_k + i\mathbb{R}} \mathcal{M}(S_1, w) x^{-w} \mathrm{d}w \\ =& \frac{x^{-1+\alpha\rho-\frac{\alpha}{2}+k}}{2\pi} \int_{\mathbb{R}} \mathcal{M}(S_1, 1 - \alpha\rho + \frac{\alpha}{2} - k + iu) x^{iu} \mathrm{d}u. \end{aligned} \tag{7.15}$$

The last equality in (7.15) follows from the change of variables $w \mapsto c_k + iu$. We observe that the identity (7.4) iterated k times gives

$$\frac{\mathcal{M}(S_1, w)}{\Gamma(w)\Gamma\left(\frac{1-w}{\alpha}\right)} = (-1)^k \left(\frac{\mathcal{M}(S_1, w+k)}{\Gamma(w+k)\Gamma(\frac{1-k-w}{\alpha})} \right) \prod_{j=1}^{k} \frac{\sin\left(\frac{\pi}{\alpha}(2-j-w)\right)}{\sin\left(\frac{\pi}{\alpha}(\alpha\rho - 2 + j + w)\right)}. \tag{7.16}$$

Further, since $\alpha\rho < 1$ we have $-1 + \alpha\rho - \frac{\alpha}{2} < 0$ and $-1 + \alpha\rho - \frac{\alpha}{2} + k > 0$ for $k \geq 2$. This implies the estimate

$$x^{-1+\alpha\rho-\frac{\alpha}{2}+k} < (1+x)^k, \quad k \geq 2,\, x > 0. \tag{7.17}$$

Combining (7.15), (7.16), and (7.17) we obtain

$$|\mathrm{e}_k(x)| < \frac{(1+x)^k}{2\pi} \int_{\mathbb{R}} \left| \mathcal{M}\left(S_1, 1 - \alpha\rho + \frac{\alpha}{2} + iu\right) \right| \times |F_1(u;k)| \times |F_2(u;k)| \, \mathrm{d}u,$$
$$k \geq 2,\, x > 0,$$

where

$$F_1(u;k) = \frac{\Gamma\left(1 - \alpha\rho + \frac{\alpha}{2} + iu - k\right) \Gamma\left(\rho - \frac{1}{2} - i\frac{u}{\alpha} + \frac{k}{\alpha}\right)}{\Gamma\left(1 - \alpha\rho + \frac{\alpha}{2} + iu\right) \Gamma\left(\rho - \frac{1}{2} - i\frac{u}{\alpha}\right)},$$

and

$$F_2(u;k) = \frac{\sin\left(\frac{\pi}{\alpha}\left(\alpha\rho - \frac{\alpha}{2} - iu + k + 1 - j\right)\right)}{\sin\left(\frac{\pi}{\alpha}\left(\frac{\alpha}{2} + iu - k - 1 + j\right)\right)}.$$

The crucial step (which is also the one that needs to be altered) is finding a bound for the function $F_2(u;k)$. From the trigonometric identities

$$\begin{aligned}\sin(x+iy) &= \sin(x)\cosh(y) + i\cos(x)\sinh(y),\\ |\sin(x+iy)|^2 &= \cosh^2(y) - \cos^2(x)\end{aligned}$$

we see that $|\sin(x)|\cosh(y) \le |\sin(x+iy)| \le \cosh(y)$, and therefore that

$$\left|\frac{\sin(a+iy)}{\sin(b+iy)}\right| \le \frac{1}{|\sin(b)|}.$$

Applying this estimate to $F_2(u;k)$ shows that

$$|F_2(u;k)| \le \prod_{j=1}^{k}\left|\csc\left(\frac{\pi}{2}\left(\frac{\alpha}{2} - k - 1 + j\right)\right)\right| = \prod_{l=1}^{k}\left|\sec\left(\frac{\pi l}{\alpha}\right)\right| =: f_\alpha(k),$$

where in the last step we have changed the index of summation so that $l = k+1-j$. Making some further estimates (here we omit a significant portion of the original proof) we end up with the following bound

$$|\mathrm{e}_k(x)| < (A(1+x))^k \times e^{-\epsilon k \ln(k)} \times f_\alpha(k)\ , \tag{7.18}$$

where $A > 0$ and $\epsilon > 0$ are constants which can depend on (α, ρ) but not on x or k. Lemma 1 in [58] then assures us that, since $\alpha \notin \mathcal{L}$, we have

$$f_\alpha(k) < B3^k, \quad k \ge 1, \tag{7.19}$$

for some constant $B = B(\alpha)$. Combining (7.18) with (7.19) implies that $\mathrm{e}_k(x) \to 0$ as $k \to +\infty$, which gives us

$$p(x) = x^{\alpha\rho-1}\lim_{k\to+\infty}\sum_{\substack{m+\alpha(n+\frac{1}{2})<k\\ m\ge 0,\, n\ge 0}} a_{m,n}x^{m+\alpha n}. \tag{7.20}$$

With some effort we can demonstrate, using (7.8) and Proposition 1 and Lemma 1 in [58], that (7.20) converges absolutely. Therefore, the order of summation in (7.20) does not matter, and the sum can be rewritten in the form (7.11). We may derive (7.10) in an analogous manner using the sequence $\hat{c}_k = 1 + k + \frac{\alpha}{2}$, $k \geq 1$. □

The assumption $\alpha \notin \mathcal{L}$ is crucial for deriving (7.19) and (7.20). The upper bound (7.19) does not hold for all irrational α, in fact, for a suitable α the function $f_\alpha(k)$ cannot be bounded by any exponential function of k. The following example illustrates this phenomenon.

Example

We define τ via its continued fraction representation

$$\tau = [a_0; a_1, a_2, \dots] = [1, 2, 2^4, 2^{1089}, \dots],$$

where the coefficients a_n are defined as $a_{n+1} = 2^{q_n^2}$, $n \geq -1$ and the numerators p_n and the denominators q_n of the n-th convergent are computed recursively via (7.12). We find the first few terms of p_n and q_n to be

$$[p_0, p_1, p_2, \dots] = [1, 3, 49, \dots], \quad \text{and} \quad [q_0, q_1, q_2, \dots] = [1, 2, 33, \dots].$$

Let us take $\alpha = 2/\tau \approx 1.34693878...$ (the interested reader may consult Proposition 1 in [58] to verify that $\alpha \in \mathcal{L}$). Since $\{a_n\}_{n\geq 1}$ are even integers, it follows from (7.12) that $\{p_n\}_{n\geq 1}$ are all odd numbers, so that we can write $p_n = 2r_n + 1$ for some integer r_n. From Theorem 13 in [62] we know that

$$|q_n\tau - p_n| < \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}} < \frac{1}{a_{n+1}} = 2^{-q_n^2},$$

and therefore that $|q_n/\alpha - r_n - 1/2| < 2^{-q_n^2}$. Using the inequality $|\cos(\pi x)| \leq \pi|x - 1/2|$, we conclude that

$$\left|\sec\left(\frac{\pi q_n}{\alpha}\right)\right| = \frac{1}{|\cos\left(\pi\left(\frac{q_n}{\alpha} - r_n\right)\right)|} \geq \frac{1}{\pi|\frac{q_n}{\alpha} - r_n - \frac{1}{2}|} > \frac{2^{q_n^2}}{\pi},$$

which shows that $f_\alpha(q_n)$ is eventually larger than C^{q_n} and even $q_n^{\epsilon q_n}$ for any $C, \epsilon > 0$. We see that in this case $|\mathrm{e}_k(x)|$ no longer vanishes as $k \to \infty$, so that the reasoning we used to derive (7.20) is also no longer valid.

As the previous example demonstrates, for certain α the terms $|\sec(\frac{\pi q_n(2/\alpha)}{\alpha})|$ become very large. That is, we get a kind of "spike" when we add the q_nth term to the product $f_\alpha(q_n)$. A natural question is: If we sample the sequence $\{f_\alpha(k)\}_{k\geq 0}$ at points directly before this spike, can the resulting subsequence be bounded by a exponential function? Remarkably, the answer is "yes". This is the content of the following lemma.

Lemma 4. *Assume that $\tau \notin \mathbb{Q}$ and $\tau > 0$. There exists a constant $C = C(\tau) > 0$ such that for all $k \geq 1$*

$$\prod_{l=1}^{q_k-1} |\sec(\pi l\tau)| \leq C6^{q_k}, \tag{7.21}$$

where $q_k = q_k(2\tau)$ is the denominator of the kth convergent for 2τ.

Proof. We use the following result (see Lemma 4 in [22] or Lemma 4 in [96]): for any $\beta > 0$, $\beta \notin \mathbb{Q}$

$$\lim_{k\to\infty} \frac{1}{q_k} \sum_{l=1}^{q_k-1} \log(2|\sin(\pi l\beta)|) = 0, \tag{7.22}$$

where $q_k = q_k(\beta)$. This is equivalent to

$$\lim_{k\to\infty} \left(\prod_{l=1}^{q_k-1} \frac{1}{|\sin(\pi l\beta)|} \right)^{\frac{1}{q_k}} = 2,$$

which implies the existence of a constant $C = C(\beta) > 0$, such that for all $k \geq 1$

$$\prod_{l=1}^{q_k-1} \frac{1}{|\sin(\pi l\beta)|} < C3^{q_k}.$$

Using the identity $\sin(\pi l\beta) = 2\sin(\pi l\frac{\beta}{2})\cos(\pi l\frac{\beta}{2})$ we conclude

$$\prod_{l=1}^{q_k-1} \frac{1}{|\cos(\pi l\frac{\beta}{2})|} < C3^{q_k}2^{q_k-1} \prod_{l=1}^{q_k-1} |\sin(\pi l\frac{\beta}{2})| < C6^{q_k}.$$

Taking $\beta = 2\tau$ we obtain (7.21). □

The work we have done to establish Lemma 4 and to sketch the proof of Theorem 35 allows

us to give a very simple proof of our main result.

Proof of Theorem 36

Lemma 4 shows that if we replace the sequence $\{c_k\}_{k\geq 1}$ in our derivation of (7.14) by the subsequence $\{c_{q_k-1}\}_{k\geq 1}$, where $q_k = q_k(2/\alpha)$, then for any irrational α the quantity $f_\alpha(q_k-1)$ is bounded by $C6^{q_k}$. Therefore $|\mathrm{e}_{q_k-1}(x)|$ vanishes as $k \to \infty$ which gives us the statement of Theorem 36 for $\alpha \in (1,2)$. The proof for $\alpha \in (0,1)$ can be obtained in a similar way using the sequence $\{\hat{c}_{q_k-1}\}_{k\geq 1}$. □

Chapter 8

Approximating Lévy processes with completely monotone jumps

Lévy processes with completely monotone jumps appear frequently in various applications of probability. In particular, all popular models used to represent log stock prices, such as VG, CGMY, and NIG processes, belong to this class. In this chapter we continue the work started in [35,60] and develop a simple yet very efficient method for approximating processes with completely monotone jumps by processes with hyper-exponential jumps. As we have seen, the latter set of processes is especially convenient for performing computations. We demonstrate, using the proposed approximation method, how to develop algorithms for pricing exotic options like Asian and barrier options for a variety of completely monotone processes. Our approximation method is based on connecting Lévy processes with completely monotone jumps with several areas of classical analysis, including Padé approximants, the Gaussian quadrature and orthogonal polynomials. We have already started to build these connections in Chapter 5; here we demonstrate their utility.

8.1 Introduction

Most researchers working in applied mathematics are familiar with the problem of choosing the right mathematical object for their modeling purpose. They must strike a balance between the tractability of their model and the model's ability to provide a realistic description of the underlying phenomenon. For example, when modeling stock prices in mathematical finance we are faced with the following dilemma: Do we choose a process which fits the empirically observed behavior of stock prices (for example, processes with jumps of infinite

activity [3,29]), or do we settle for a simpler model which provides us with explicit formulas and efficient numerical algorithms? The first choice leads us to the most popular Lévy processes used in finance such as the VG, CGMY, and NIG processes. These processes, which belong to a wider class of processes with completely monotone jumps, provide a good fit for market data, and they are flexible enough to accommodate such desirable features as jumps of infinite activity and finite or infinite variation. They also enjoy a certain degree of analytical tractability (for example, VG and NIG processes have explicit transition densities), and European option prices and Greeks can be computed quite easily. However, the computation of more exotic option prices (such as barrier, lookback and Asian options) is a much more challenging task. On the other hand, hyper-exponential processes, and more generally processes with jumps of rational transform, form the most convenient class for performing numerical calculations. This is due to the fact that for these processes we have explicit Wiener-Hopf factorizations (Theorem 8) and explicit formulas for the Mellin transform of the exponential functional (4.14) which lead to simple and efficient numerical algorithms for pricing barrier options (Section 3.3), look-back options [26] and Asian options (Chapter 6). As we have remarked several times, the major disadvantage of processes with jumps of rational transform is that they are finite activity processes, and this type of behaviour seems to be incompatible with empirical results [3,29] from the stock market.

A natural way to reconcile our two competing objectives is to approximate processes with completely monotone jumps with hyper-exponential processes. Two approximations of this sort have been developed recently: Jeannin and Pistorius [60] use least squared optimization in order to find the approximating hyper-exponential process, while Crosby, Le Saux and Mijatović [35] use a more direct approach based on the Gaussian quadrature. The goal of this chapter is to present a new method for approximating Lévy processes with completely monotone jumps, and to demonstrate that this method is natural, simple and very efficient.

The key idea behind the approach presented here is that approximating a Lévy process X is equivalent to finding an approximation of its Laplace exponent $\psi(z)$. The Laplace exponent of a hyper-exponential processes is a rational function, therefore our problem reduces to two steps: (i) finding a good rational approximation $\tilde{\psi}(z) \approx \psi(z)$; and (ii) ensuring that the rational function $\tilde{\psi}(z)$ is itself a Laplace exponent of a hyper-exponential process $\tilde{X}$.

Point (ii) helps us to immediately narrow our selection process. Let $r(z)$ be a rational function with numerator $P(z)$ and denominator $Q(z)$, and define the ***degree*** of $r(z)$ to be $\deg(r) := \max\{\deg(P), \deg(Q)\}$. Further, suppose that $\deg(P) = m$ and $\deg(Q) = n$. It

is known (see Proposition 2 in [14]) that the Laplace exponent of a Lévy process satisfies $\psi(iz) = O(z^2)$ as $z \to \infty$, therefore $r(z)$ cannot be the Laplace exponent of a Lévy process if $m > n + 2$. If $m < n$ then necessarily $r(z) \to 0$ as $z \to \infty$, and we can show that a rational function with this property cannot be the Laplace exponent of a Lévy process Y – unless $Y \equiv 0$ almost surely. We conclude that our rational approximation $r(z)$ must satisfy: a) $\deg(P) = \deg(Q)$; b) $\deg(P) = \deg(Q) + 1$; or c) $\deg(P) = \deg(Q) + 2$ in order to qualify as the Laplace exponent of a Lévy process.

We recall that we have already seen one solution to our problem in Section 5.3.2. This relied on the theory of rational approximations of Pick functions, and on solving the Cauchy interpolation problem. Although this is a valid and useful method, it also has a number of limitations. First, we are forced to solve a linear system to get our solution, second we do not know whether, and at what rate, our rational approximation $r(z)$ converges to $\psi(z)$ as $\deg(r) \to \infty$, and third we do not have any control over whether the approximating process is a pure jump process, a process with drift, or a process with drift and Gaussian component.

In this chapter we show that if we properly employ the tools from Chapter 5 we can develop an analogous method using the theory of Padé approximants of Stieltjes functions which remedies these shortcomings. We will show that a Padé approximant of $\psi(z)$ exists, and must be the Laplace exponent of a hyper-exponential process. From our discussion above, and using the notation of Chapter 5, this is equivalent to the statement that at least one of

$$\psi^{[n/n]}(z), \quad \psi^{[n+1/n]}(z), \quad \text{or} \quad \psi^{[n+2/n]}(z) \tag{8.1}$$

exists and is the Laplace exponent of a hyper-exponential process. We show that we can always frame the problem in terms of the Gaussian quadrature, and that in the most important cases we can use the connection with the Jacobi polynomials to avoid solving linear systems. We use the theory of Padé approximants of Stieltjes functions to demonstrate convergence (as $n \to \infty$) and discover the rate of convergence. Finally, we demonstrate that the proposed method gives much more control over the approximating process. That is, if X is a one-sided process, then both an approximating process with Gaussian component and one without are possible.

This chapter is organized as follows. Section 8.2 contains the main results on approximating Lévy processes with completely monotone jumps (treating the two-sided and one-sided cases separately). Section 8.3 discusses the important special cases of the gamma subordinator

and of the one-sided generalized tempered stable process; in both cases the Padé approximant is given by an explicit formula (i.e. there is no need to solve a linear system). In this section we also discuss how to use these results to construct explicit approximations of the Laplace exponents of VG, CGMY and NIG processes, and we discuss some extensions of our approximation scheme. In Section 8.4 we present the results of several numerical experiments which demonstrate the efficiency of the approximation method. We compute the Lévy density, the CDF, and the prices of various options for the approximating processes and investigate numerical convergence. Finally, in Section 8.5 we compare the new technique with the existing methods that inspired this research [35, 60].

8.2 Main results

In this chapter we will work exclusively with processes which are completely monotone. We further assume that the Lévy densities of our processes, which we recall are defined by

$$\pi(x) = \mathbb{I}(x < 0) \int_{\mathbb{R}^-} e^{-ux} \mu(\mathrm{d}u) + \mathbb{I}(x > 0) \int_{\mathbb{R}^+} e^{-ux} \mu(\mathrm{d}u), \tag{8.2}$$

decay exponentially as $|x| \to 0$. This mild assumption, which holds for the vast majority of completely monotone processes used in mathematical finance, has three consequences. The first, is that we may use the cut-off function $h(x) \equiv 1$ in our representation of the Laplace exponent

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + \int_{\mathbb{R}} \left(e^{zx} - 1 - zxh(x)\right) \pi(x) \mathrm{d}x, \tag{8.3}$$

for any process under discussion. The second, is that integrability condition (2.21) on the representing measure $\mu(\mathrm{d}u)$ can be expressed more simply as

$$\int_{\mathbb{R}} x^2 \pi(x) \mathrm{d}x < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}} |u|^{-3} \mu(\mathrm{d}u) < \infty, \text{ and} \tag{8.4}$$

$$\int_{\mathbb{R}} |x| \pi(x) \mathrm{d}x < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}} u^{-2} \mu(\mathrm{d}u) < \infty. \tag{8.5}$$

Finally, the third, is that the quantities

$$\rho := \sup\left\{c \geq 0 : \int_{\mathbb{R}^+} e^{cx}\pi(x)\mathrm{d}x < \infty\right\} = \sup\left\{u \geq 0 : \mu((0,u)) = 0\right\}, \text{ and}$$
$$\hat{\rho} := \sup\left\{c \geq 0 : \int_{\mathbb{R}^-} e^{-cx}\pi(x)\mathrm{d}x < \infty\right\} = \sup\left\{u \geq 0 : \mu((-u,0)) = 0\right\}$$

are strictly positive. We will continue to use the notation of Section 5.3.1 and denote by $\mathcal{CM}(\hat{\rho},\rho)$ the class of Lévy processes with completely monotone jumps and parameters ρ and $\hat{\rho}$.

We recall that one of the simplest completely monotone processes is the hyper-exponential process. Setting

$$\mu(\mathrm{d}x) = \sum_{n=1}^{\hat{N}} \hat{a}_n\hat{\rho}_n\delta_{-\hat{\rho}_n}(\mathrm{d}x) + \sum_{n=1}^{N} a_n\rho_n\delta_{\rho_n}(\mathrm{d}x),$$

and using (8.2), we get the Lévy density

$$\pi(x) = \mathbb{I}(x<0)\sum_{n=1}^{\hat{N}} \hat{a}_n\hat{\rho}_n e^{\hat{\rho}_n x} + \mathbb{I}(x>0)\sum_{n=1}^{N} a_n\rho_n e^{-\rho_n x} \tag{8.6}$$

of a hyper-exponential process. By plugging this into (8.3) we get the Laplace exponent

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z\sum_{n=1}^{N}\frac{a_n}{\rho_n - z} - z\sum_{n=1}^{\hat{N}}\frac{\hat{a}_n}{\hat{\rho}_n + z} \tag{8.7}$$

of a hyper-exponential process defined by generating triple $(a,\sigma^2,\pi)_{h\equiv 0}$, or the Laplace exponent

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z^2\sum_{n=1}^{\hat{N}}\frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} + z^2\sum_{n=1}^{N}\frac{a_n}{\rho_n(\rho_n - z)} \tag{8.8}$$

of a hyper-exponential process defined by generating triple $(a,\sigma^2,\pi)_{h\equiv 1}$.

Before continuing, the reader may wish to revisit Section 5.3.2, in particular the results concerning Padé approximants, the Gaussian quadrature, and orthogonal polynomials.

8.2.1 Approximating Lévy processes with two-sided jumps

For $X \in \mathcal{CM}(\hat{\rho}, \rho)$ we define

$$\mu^*(A) = \mu(\{v \in \mathbb{R} \ : \ v^{-1} \in A\}), \quad A \in \mathcal{B}_{\mathbb{R}}, \tag{8.9}$$

where $\mu(\mathrm{d}u)$ is the representing measure. Note that $\mathrm{supp}(\mu^*) \subseteq [-1/\hat{\rho}, 1/\rho]$, and if the measure $\mu(\mathrm{d}u)$ is absolutely continuous with a density $m(u)$, then $\mu^*(\mathrm{d}v)$ also has a density, which is given by

$$m^*(v) = |v|^{-2} m(v^{-1}).$$

The measure $\mu^*(\mathrm{d}v)$ will play a central role in this chapter. The next Lemma reveals the following important properties of $\mu^*(\mathrm{d}v)$: a) $|v|^2\mu^*(\mathrm{d}v)$ is a finite measure when X is a finite variation process; and b) $|v|^3\mu^*(\mathrm{d}v)$ is always finite.

Lemma 5. *Assume that $X \in \mathcal{CM}(\hat{\rho}, \rho)$. Then*

$$\int_{[-1/\hat{\rho}, 1/\rho]} |v|^3 \mu^*(\mathrm{d}v) < \infty,$$

and

$$\int_{[-1/\hat{\rho}, 1/\rho]} |v|^2 \mu^*(\mathrm{d}v) < \infty \iff \text{if and only if } X \text{ has jumps of finite variation.}$$

Proof. The result follows from (8.4) and (8.5) by a change of variables $u \mapsto 1/v$. □

Now we introduce our first approximation. We start with $X \in \mathcal{CM}(\hat{\rho}, \rho)$ defined by the characteristic triple $(a, 0, \pi)_{h \equiv 1}$. Note that the process X has zero Gaussian component, but we lose no generality by this assumption. If we know how to approximate a Lévy processes with zero Gaussian component, we know how to approximate a general Lévy process: we can always add a scaled Brownian motion to our hyper-exponential approximation.

According to Lemma 5, $|v|^3\mu^*(\mathrm{d}v)$ is a finite measure on the interval $[-1/\hat{\rho}, 1/\rho]$, and if we assume that X is not already a hyper-exponential process (this will be a standing assumption from now on), then $|v|^3\mu^*(\mathrm{d}v)$ has infinite support. Therefore, we can define the Gaussian quadrature with respect to $|v|^3\mu^*(\mathrm{d}v)$. Accordingly, let $\{x_i\}_{1 \leq i \leq n}$ and $\{w_i\}_{1 \leq i \leq n}$

be the nodes and the weights of the Gaussian quadrature of order n with respect to this measure, and define the function

$$\psi_n(z) := az + z^2 \sum_{i=1}^{n} \frac{w_i}{1 - zx_i}. \tag{8.10}$$

Theorem 37. *Let $X \in \mathcal{CM}(\hat{\rho}, \rho)$ with Laplace exponent $\psi(z)$.*

(i) The function $\psi_n(z)$ is the $[n+1/n]$ Padé approximant of $\psi(z)$.

(ii) The function $\psi_n(z)$ is the Laplace exponent of a hyper-exponential process $X^{(n)}$ having the characteristic triple $(a, \sigma_n^2, \pi_n)_{h\equiv 1}$, where

$$\sigma_n^2 := \begin{cases} 0, & \text{if } \ x_i \neq 0 \ \text{ for all } \ 1 \leq i \leq n, \\ 2w_j & \text{if } \ x_j = 0 \ \text{ for some } \ 1 \leq j \leq n, \end{cases} \tag{8.11}$$

and

$$\pi_n(x) := \begin{cases} \displaystyle\sum_{1\leq i\leq n \,:\, x_i<0} w_i |x_i|^{-3} e^{-\frac{x}{x_i}}, & \text{if } \ x < 0, \\ \displaystyle\sum_{1\leq i\leq n \,:\, x_i>0} w_i x_i^{-3} e^{-\frac{x}{x_i}}, & \text{if } \ x > 0. \end{cases} \tag{8.12}$$

If one of the sums in (8.12) is empty, it should be interpreted as zero.

(iii) The random variables $X_1^{(n)}$ and X_1 satisfy $\mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j]$ for $1 \leq j \leq 2n+1$.

Proof. Our first goal is to establish an integral representation of $\psi(z)$ in terms of the measure $\mu^*(\mathrm{d}v)$. Initially, we assume that $z \in \mathbb{C}$ with $-\hat{\rho} < \mathrm{Re}(z) < \rho$. We substitute (8.2) into (8.3), use Fubini's theorem to interchange the order of integration, and obtain

$$\psi(z) = az + z^2 \int_{\mathbb{R}} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2}. \tag{8.13}$$

Performing a change of variables $u \mapsto 1/v$ in the above integral and using the fact that $\mu((-\hat{\rho}, \rho)) = 0$ formula (8.13) becomes

$$\psi(z) = az + z^2 \int\limits_{[-1/\hat{\rho}, 1/\rho]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz}, \quad -\hat{\rho} < \mathrm{Re}(z) < \rho. \tag{8.14}$$

By analytic continuation we can see that the above formula is valid in a larger region $\mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$.

Now, let us prove (i). By definition, the Gaussian quadrature of order n is exact for polynomials of degree $\leq 2n-1$, therefore

$$\int_{[-1/\hat{\rho},1/\rho]} v^k |v|^3 \mu^*(\mathrm{d}v) = \sum_{i=1}^{n} x_i^k w_i, \quad k = 0, 1, 2, \ldots, 2n-1.$$

The above identity is equivalent to the identity

$$\left(\frac{\mathrm{d}^k}{\mathrm{d}z^k} \int_{[-1/\hat{\rho},1/\rho]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1-vz} \right) \Bigg|_{z=0} = \left(\frac{\mathrm{d}^k}{\mathrm{d}z^k} \sum_{i=1}^{n} \frac{w_i}{1-zx_i} \right) \Bigg|_{z=0}, \tag{8.15}$$
$$k = 0, 1, 2, \ldots, 2n-1,$$

which, together with (8.10) and (8.14), implies that

$$\psi^{(k)}(0) = \psi_n^{(k)}(0), \quad k = 0, 1, 2, \ldots, 2n+1. \tag{8.16}$$

By definition $\psi_n(z)$ is a rational function, which can be written in the form $P(z)/Q(z)$ with $\deg(P) \leq n+1$ and $\deg(Q) = n$. Using this fact, the definition of the Padé approximant (Section 5.3.2), and (8.16), we see that $\psi_n(z) \equiv \psi^{[n+1/n]}(z)$, which proves (i).

To prove (ii) we just need to rearrange (8.10) so that we recognize the Laplace exponent of a hyper-exponential process. For a node x_i of our Gaussian quadrature we may do some straight-forward algebra to show that

$$\frac{w_i}{1-zx_i} = \begin{cases} \frac{x_i^{-2} w_i}{x_i^{-1}(x_i^{-1}-z)} & x_i > 0 \\ w_i & x_i = 0 \\ \frac{x_i^{-2} w_i}{|x_i^{-1}|(|x_i^{-1}|-z)} & x_i < 0 \end{cases}.$$

Now we set

$$\begin{aligned} &a_i = x_i^{-2} w_i, \text{ and } \rho_i = x_i^{-1} && \text{if } x_i > 0, \\ &\sigma^2 = 2w_i && \text{if } x_i = 0, \text{ and} \\ &\hat{a}_i = x_i^{-2} w_i, \text{ and } \hat{\rho}_i = |x_i^{-1}| && \text{if } x_i < 0, \end{aligned}$$

and compare with (8.8) and (8.6) to get the result.

Let $X^{(n)}$ denote the hyper-exponential process corresponding to $\psi_n(z)$. Formula (8.16) shows that the first $2n+1$ cumulants of $X_1^{(n)}$ are equal to the corresponding cumulants of X_1, which is equivalent to the equality of the moments and proves item (iii). □

The next important question is how fast the sequence of approximations $\{\psi_n(z)\}_{n\geq 1}$ converges to $\psi(z)$. As we saw in the proof of Theorem 37 (see also [74,101]), the Laplace exponent $\psi(z)$ of a process $X \in \mathcal{CM}(\hat{\rho}, \rho)$ is analytic in the cut complex plane $\mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$. In Theorem 38 we will establish that the sequence $\{\psi_n(z)\}_{n\geq 1}$ converges to $\psi(z)$ *everywhere* in this region, and the convergence is exponentially fast on compact subsets of $\mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$. This behavior should be compared with Taylor series approximations, which converge only in a circle of finite radius (lying entirely in the region of analyticity of $\psi(z)$). We see that Padé approximants are very well suited to approximate Laplace exponents of processes in $\mathcal{CM}(\hat{\rho}, \rho)$.

From Theorem 28 we know that Padé approximants of Stieltjes functions converge exponentially fast on compact sets in the domain of the original function. We also know from Theorem 23 that the Laplace exponent $\psi(z)$ of a process $X \in \mathcal{CM}(\hat{\rho}, \rho)$ can be expressed in terms of a Stieltjes function $g(z)$ in the following manner

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + \frac{z^2}{1 + \frac{z}{\hat{\rho}}} g\left(-\frac{z}{1 + \frac{z}{\hat{\rho}}}\right). \tag{8.17}$$

We will make use of these two theorems to prove Theorem 38. However, before using the representation (8.17), we must prove it, since we did not do so in Chapter 5.

Theorem 23 (once more). *Assume X is a Lévy process with Laplace exponent $\psi(z)$ and $\hat{\rho}$, $\rho > 0$. The following assertions are equivalent:*

(i) $X \in \mathcal{CM}(\hat{\rho}, \rho)$.

(ii) $\psi(z)$ *has the form* (8.17) *where* $a, \sigma \in \mathbb{R}$, *and* $g(z)$ *is a Stieltjes function with radius of convergence* $R = (1/\rho + 1/\hat{\rho})^{-1}$.

Proof. To prove $(i) \Rightarrow (ii)$ we remark (by the same reasoning used to establish (8.14)) that

we may write the $(a, \sigma^2, \pi)_{h\equiv 1}$ representation of $\psi(z)$ as

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + z^2 \int\limits_{[-1/\hat{\rho}, 1/\rho]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz}. \tag{8.18}$$

Let us denote $|v|^3\mu^*(\mathrm{d}v)$ by $\eta(\mathrm{d}v)$ and define

$$g(z) := \int\limits_{(0, \frac{1}{\hat{\rho}} + \frac{1}{\rho}]} \frac{\eta(\mathrm{d}(u - \frac{1}{\hat{\rho}}))}{1 + uz}, \quad \text{and} \quad f(z) := zg(z). \tag{8.19}$$

We observe that $g(z)$ is a Stieltjes function with the radius of convergence $R = (1/\rho + 1/\hat{\rho})^{-1}$. Changing the variable of integration $v \mapsto u - \frac{1}{\hat{\rho}}$ in (8.19) we obtain

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az - zf\left(-\frac{z}{1 + \frac{z}{\hat{\rho}}}\right) = \frac{\sigma^2 z^2}{2} + az + \frac{z^2}{1 + \frac{z}{\hat{\rho}}} g\left(-\frac{z}{1 + \frac{z}{\hat{\rho}}}\right), \tag{8.20}$$

which proves $(i) \Rightarrow (ii)$. To prove the converse, we just reverse our steps. □

With this task completed, establishing the proof of the following theorem is just a matter of combining the right results from Chapter 5.

Theorem 38. *Let $\psi(z)$ and $\psi_n(z)$ be defined as in Theorem 37. For any compact set $A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$ there exist $c_1 = c_1(A) > 0$ and $c_2 = c_2(A) > 0$ such that for all $z \in A$ and all $n \geq 1$*

$$|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.$$

Proof. First we write $\psi(z)$ as in (8.20) with $\sigma = 0$ and $f(z)$, $g(z)$, and R defined as in (8.19). Further, we define $w := -z/(1 + z/\hat{\rho})$ and $F(z) := f(w)$. According to Theorem 29, the $[n/n]$ Padé approximant is invariant under rational transformations of the variable. Therefore, $F^{[n/n]}(z) = f^{[n/n]}(w)$ provided $f^{[n/n]}(z)$ exists. Theorem 31 shows that $f^{[n/n]}(z) = zg^{[n-1/n]}(z)$, and we know that $g^{[n-1,n]}(z)$ exists from Theorem 25 and from the fact that $g(z)$ is a Stieltjes function; therefore, $f^{[n/n]}(z)$ and $F^{[n/n]}(z)$ also exist. Now using (8.20) and applying Theorem 31 again, we have $z^{-1}(\psi^{[n+1/n]}(z) - az) = F^{[n/n]}(z)$. Therefore, we may conclude that

$$\psi_n(z) = \psi^{[n+1,n]}(z) = az - zf^{[n/n]}\left(-\frac{z}{1 + \frac{z}{\hat{\rho}}}\right) = az + \frac{z^2}{1 + \frac{z}{\hat{\rho}}} g^{[n-1/n]}\left(-\frac{z}{1 + \frac{z}{\hat{\rho}}}\right), \tag{8.21}$$

where the first equality follows from the results of Theorem 37. Now, from Theorem 28 we know that the functions $g^{[n-1/n]}(z)$ converge to $g(z)$ exponentially fast on compact subsets of $\mathbb{C}\setminus(-\infty,-R]$. It is easy to see that the function $w(z)=-z/(1+z/\hat{\rho})$ maps compact subsets of $\mathbb{C}\setminus\{(-\infty,-\hat{\rho}]\cup[\rho,\infty)\}$ onto compact subsets of $\mathbb{C}\setminus(-\infty,-R]$. This fact combined with (8.20) and (8.21) ends the proof of Theorem 38. □

The results of Theorem 37 show that the Padé approximant $\psi^{[n+1/n]}(z)$ is always a Laplace exponent of a hyper-exponential process. However, as we discussed in the Introduction, there are two other Padé approximants, $\psi^{[n/n]}(z)$ and $\psi^{[n+2/n]}(z)$, which can also qualify as Laplace exponents. While we do not have a counterexample, we believe that in general it is not true that for all Lévy processes $X\in\mathcal{CM}(\hat{\rho},\rho)$ the functions $\psi^{[n/n]}(z)$ and $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyper-exponential processes. However, more can be said for processes with one-sided jumps, and we present these results in the next section.

8.2.2 Approximating Lévy processes with one-sided jumps

In this section we will consider two cases: when the process X has (i) jumps of finite variation or (ii) jumps of infinite variation. In the first case it is enough to consider subordinators with zero linear drift. If we know how to approximate such subordinators, we can always add a linear drift and a Gaussian component later. The following theorem is the analogy of Theorem 37 for subordinators. The key difference is that we have three (rather than one) approximations to choose from.

Theorem 39. *Assume that $X\in\mathcal{C}M(+\infty,\rho)$ is a subordinator defined by the characteristic triple $(0,0,\pi)_{h\equiv 0}$. Let $\psi(z)$ denote the Laplace exponent of X, and fix $k\in\{0,1,2\}$.*

(i) Let $\{x_i\}_{1\le i\le n}$ and $\{w_i\}_{1\le i\le n}$ be the nodes and the weights of the Gaussian quadrature with respect to the measure $v^{2+k}\mu^*(\mathrm{d}v)$. Then

$$\psi^{[n+k/n]}(z)=\sum_{j=1}^{k}\psi^{(j)}(0)\frac{z^j}{j!}+z^{k+1}\sum_{i=1}^{n}\frac{w_i}{1-zx_i}. \tag{8.22}$$

(ii) The function $\psi^{[n+k/n]}(z)$ is the Laplace exponent of a hyper-exponential process $X^{(n)}$. The process $X^{(n)}$ has a Lévy measure with density

$$\pi_n(x):=\mathbb{I}(x>0)\sum_{i=1}^{n}w_i x_i^{-2-k}e^{-\frac{x}{x_i}}, \tag{8.23}$$

and is defined by the characteristic triple

$$\begin{cases} (0, 0, \pi_n)_{h\equiv 0} & \text{if } \ k = 0, \\ (\psi'(0) - \sum_{i=1}^n w_i/x_i, 0, \pi_n)_{h\equiv 0} & \text{if } \ k = 1, \\ (\psi'(0), \psi''(0) - 2\sum_{i=1}^n w_i/x_i, \pi_n)_{h\equiv 1}, & \text{if } \ k = 2. \end{cases} \tag{8.24}$$

The process $X^{(n)}$ is a subordinator if $k = 0$ or $k = 1$ (with zero linear drift in the former case and positive linear drift in the latter case), and $X^{(n)}$ is a spectrally positive process with a non-zero Gaussian component if $k = 2$.

(iii) The functions $\psi^{[n+k/n]}(z)$ converge to $\psi(z)$ exponentially fast on compact subsets of $\mathbb{C} \setminus [\rho, \infty)$.

Before proving Theorem 39, we need to establish the following auxiliary result. This lemma will help us establish that $\psi'(0) - \sum_{1\le i\le n} w_i/x_i$ and $\psi''(0) - 2\sum_{1\le i\le n} w_i/x_i$ are both positive. In other words, it will help us show that the process resulting from our approximation when $k = 1$ is a subordinator, and that the process resulting from the $k = 2$ approximation is a Lévy process.

Lemma 6. *Assume that $\nu(\mathrm{d}x)$ is a finite positive measure on $(0, R]$. Let $\{x_i\}_{1\le i\le n}$ and $\{w_i\}_{1\le i\le n}$ be the nodes and the weights of the Gaussian quadrature with respect to the measure $x\nu(\mathrm{d}x)$ on $(0, R]$. Then*

$$\sum_{i=1}^n \frac{w_i}{x_i} < \int_{(0,R]} \nu(\mathrm{d}x).$$

Proof. Consider two Stieltjes functions

$$f(z) := \int_{(0,R]} \frac{\nu(\mathrm{d}x)}{1 + xz}, \quad \text{and} \quad g(z) := \int_{(0,R]} \frac{x\nu(\mathrm{d}x)}{1 + xz}.$$

It is easy to check that $f(z) = f(0) - zg(z)$. From Theorems 27 and 31 in the Chapter 5 we find that $f^{[n/n]}(z) = f(0) - zg^{[n-1/n]}(z)$ and $g^{[n-1/n]}(z) = \sum_{1\le i\le n} w_i/(1 + x_i z)$. Therefore

$$\lim_{z\to+\infty} f^{[n/n]}(z) = f(0) - \sum_{i=1}^n \frac{w_i}{x_i}. \tag{8.25}$$

Consider the function $F(z) := (f(0)/f(z) - 1)/z$. Note that $F(z) \to -f'(0)/f(0)$ as $z \to 0$, and that $F(z)$ is analytic in some neighborhood of zero. From Theorems 30 and 31 in the

Chapter 5 we obtain

$$F^{[n-1/n]}(z) = \frac{1}{z}\left(\frac{f(0)}{f^{[n/n]}(z)} - 1\right),$$

which can be rewritten as

$$f^{[n/n]}(z) = \frac{f(0)}{1 + zF^{[n-1/n]}(z)}. \tag{8.26}$$

Theorem 1.3 in [61] tells us that $F(z)$ is also a Stieltjes function, and since it is analytic in the neighborhood of zero, it has a positive radius of convergence (and therefore, finite moments). Theorem 27 in Chapter 5 implies that $\lim_{z\to+\infty} zF^{[n-1/n]}(z)$ is finite and positive. This fact combined with (8.26) shows that $\lim_{z\to+\infty} f^{[n/n]}(z)$ is strictly positive, and applying (8.25) we obtain the statement of the lemma. □

Proof of Theorem 39

First, we note that since the process X has jumps of finite variation, Lemma 5 ensures that $v^2\mu^*(\mathrm{d}v)$ is a finite measure. Then, (8.2) and (8.3) give us

$$\psi(z) = z\int_{(0,\frac{1}{\rho}]} \frac{v^2\mu^*(\mathrm{d}v)}{1-vz}. \tag{8.27}$$

We will prove only the case $k = 2$; the other two cases can be treated in the same way. We start with the identity (8.27) and rewrite it in the equivalent form

$$\begin{aligned}\psi(z) &= z\int_{(0,\frac{1}{\rho}]} v^2\mu^*(\mathrm{d}v) + z^2\int_{(0,\frac{1}{\rho}]} v^3\mu^*(\mathrm{d}v) + z^3\int_{(0,\frac{1}{\rho}]} \frac{v^4\mu^*(\mathrm{d}v)}{1-vz}\\ &= \psi'(0)z + \psi''(0)\frac{z^2}{2} + z^3\int_{(0,\frac{1}{\rho}]} \frac{v^4\mu^*(\mathrm{d}v)}{1-vz}.\end{aligned} \tag{8.28}$$

The result of item (i) follows the above expression and Theorems 27 and 31 in Chapter 5.

Next, let us prove (ii). We use Lemma 6, from which it follows that

$$\frac{1}{2}\psi''(0) - \sum_{i=1}^{n} w_i/x_i = \int_{(0,\frac{1}{\rho}]} v^3\mu^*(\mathrm{d}v) - \sum_{i=1}^{n} w_i/x_i > 0,$$

and thus the Gaussian component is positive. From (8.8) we know the Laplace exponent of the process $X^{(n)}$ corresponding to the characteristic triple
$(\psi'(0), \psi''(0) - 2\sum_{1\le i\le n} w_i/x_i, \pi_n)_{h\equiv 1}$ is

$$\begin{aligned}\psi_{X^{(n)}}(z) &= \left(\psi''(0) - \sum_{i=1}^{n} \frac{w_i}{x_i}\right)\frac{z^2}{2} + \psi'(0)z + z^2\sum_{i=1}^{n}\frac{w_i}{x_i(1-zx_i)} \\ &= \psi'(0)z + \psi''(0)\frac{z^2}{2} + z^3\sum_{i=1}^{n}\frac{w_i}{1-zx_i} = \psi^{[n+2/n]}(z).\end{aligned}$$

This proves (ii). Item (iii) follows from (8.27) and Theorem 28 in Chapter 5. □

Next we consider the second class of processes with one-sided jumps: spectrally positive processes with jumps of infinite variation. Again, without loss of generality we assume that there is no Gaussian component. Our results are presented in the following theorem (the proof is omitted, as it is identical to the proof of Theorem 39).

Theorem 40. *Assume that $X \in \mathcal{C}M(+\infty, \rho)$ is a spectrally positive process having jumps of infinite variation defined by the characteristic triple $(a, 0, \pi)_{h\equiv 1}$. Let $\psi(z)$ denote the Laplace exponent of X, and fix $k \in \{1, 2\}$.*

(i) Let $\{x_i\}_{1\le i\le n}$ and $\{w_i\}_{1\le i\le n}$ be the nodes and the weights of the Gaussian quadrature with respect to the measure $v^{2+k}\mu^*(\mathrm{d}v)$. Then

$$\psi^{[n+k/n]}(z) = \sum_{j=1}^{k}\psi^{(j)}(0)\frac{z^j}{j!} + z^{k+1}\sum_{i=1}^{n}\frac{w_i}{1-zx_i}. \tag{8.29}$$

(ii) The function $\psi^{[n+k/n]}(z)$ is the Laplace exponent of a hyper-exponential process $X^{(n)}$. The process $X^{(n)}$ has a Lévy measure with density,

$$\pi(x) := \mathbb{I}(x > 0)\sum_{i=1}^{n} w_i x_i^{-2-k} e^{-\frac{x}{x_i}},$$

and is defined by the characteristic triple

$$\begin{cases} (\psi'(0), 0, \pi)_{h\equiv 1} & \text{if } k = 1, \\ (\psi'(0), \psi''(0) - 2\sum_{i=1}^{n} w_i/x_i, \pi)_{h\equiv 1}, & \text{if } k = 2. \end{cases} \tag{8.30}$$

(iii) The functions $\psi^{[n+k/n]}(z)$ converge to $\psi(z)$ exponentially fast on compact subsets of $\mathbb{C} \setminus [\rho, \infty)$.

Remark

Let us examine why we have three different approximations in the case of subordinators and only two approximations in the case of spectrally positive processes. This happens because in the case of spectrally positive processes with jumps of infinite variation the measure $v^2\mu^*(\mathrm{d}v)$ is not finite (see Lemma 5), therefore we cannot define the Gaussian quadrature with respect to this measure and our method of proving that $\psi^{[n/n]}(z)$ is a Laplace exponent (in Theorem 39) will not work. While we do not have a counterexample, we believe that it is not true that for any spectrally positive process $X \in \mathcal{CM}(+\infty, \rho)$ with completely monotone jumps (and Laplace exponent $\psi(z)$) the function $\psi^{[n/n]}(z)$ is the Laplace exponent of a hyper-exponential process.

8.3 Explicit examples and extensions of the algorithm

In this section we pursue three goals. First, we show how the results of Theorems 39 and 40 lead to explicit formulas for Gamma subordinators and one-sided generalized tempered stable processes. Then we use these results to construct explicit hyper-exponential approximations to VG, CGMY and NIG processes. Finally, we discuss two extensions of the approximation technique described in the previous section.

Example 1: Gamma subordinator

Consider the familiar Gamma subordinator with Laplace exponent $\psi(z) = -\log(1-z)$. The following proposition gives explicit formulas for the approximations described in Theorem 39. The notation $P_n^{(\alpha,\beta)}(x)$ refers to the nth Jacobi polynomial with parameters α and β (see Section 5.3.2 for a definition.)

Proposition 3. *Let X be a Gamma process defined by the Laplace exponent $\psi(z) = -\log(1-z)$. Fix $k \in \{0, 1, 2\}$.*

(i) The denominators of the Padé approximants $\psi^{[n+k,k]}(z) = p_{n,k}(z)/q_{n,k}(z)$ are given by

$$q_{n,k}(z) = Cz^n P_n^{(0,k)}\left(\tfrac{2}{z} - 1\right), \tag{8.31}$$

where $C = C(n,k)$ is a constant determined by the condition $q_{n,k}(0) = 1$. In the case $k = 0$ the numerators are also given by an explicit formula

$$p_{n,0}(z) = 2\sum_{j=0}^{n} \binom{n}{j}^2 (H_{n-j} - H_j)\,(1-z)^j, \tag{8.32}$$

where $H_0 := 0$ *and* $H_j := 1 + 1/2 + \cdots + 1/j$ *for* $j \geq 1$.

(ii) The nodes of the Gaussian quadrature described in Theorem 39 are given by $x_i = (y_i+1)/2$, *where* $y_i \in (-1, 1)$ *are the roots of the Jacobi polynomials* $P_n^{(0,k)}(y)$.

Proof. We recall from our example in Section 5.3.3 that

$$-\log(1-z) = z\int_0^1 \frac{\mathrm{d}v}{1-zv},$$

and so the measure $v^2\mu^*(\mathrm{d}v)$ is just the Lebesgue measure on $(0, 1)$. From Section 5.3.2 we know that the orthogonal polynomials with respect to the measure $\mathbb{I}(0 < x < 1)v^k\mathrm{d}v$ are given by the shifted Jacobi polynomials $P_n^{(0,k)}(2z-1)$ (in fact, when $k = 0$ we get the Legendre polynomials). Formula (8.31) follows from this fact and Theorems 39 and 27.

Statement (ii) follows from the fact that the nodes of the Gaussian quadrature coincide with the roots of orthogonal polynomials, i.e. Theorem 26.

To derive the explicit expression for $p_{n,0}(z)$ in (8.32), we use a known formula for the numerator of the $[n/n]$ Padé approximant (at $z = 1$) of the natural logarithm. The formula ((5) in [112]) is given in terms of the digamma function $\psi(z)$ as

$$2\sum_{j=0}^{n}\binom{n}{j}^2 \left(\psi(n-j+1) - \psi(j+1)\right) z^j.$$

To get (8.32) we use the well-known identity for the digamma function

$$\psi(n) = H_{n-1} - \gamma, \quad n \in \mathbb{N},$$

where γ is the Euler-Mascheroni constant (Formula 6.3.2 in [2]). □

Example 2: Tempered stable subordinator/spectrally positive process

Consider a Lévy process X defined by the Laplace exponent

$$\psi(z) = \Gamma(-\alpha)((1-z)^\alpha - 1), \tag{8.33}$$

where $\alpha \in (0, 1) \cup (1, 2)$. Comparing (8.33) with (2.27) we see that X is a generalized tempered stable process with only positive jumps. When $\alpha \in (0, 1)$ the process is a subordinator

with zero linear drift, and when $\alpha \in (1,2)$ it is a spectrally positive process with jumps of infinite variation and zero Gaussian component.

Proposition 4. *Let X be a tempered stable process defined by the Laplace exponent (8.33). For $\alpha \in (0,1)$ (resp. $\alpha \in (1,2)$) we fix a value of $k \in \{0,1,2\}$ (resp. $k \in \{1,2\}$).*

(i) The denominators and the numerators of the Padé approximants $\psi^{[n+k/n]}(z) = p_{n,k}(z)/q_{n,k}(z)$ are given by

$$q_{n,k}(z) = Cz^n P_n^{(\alpha,k-\alpha)}\left(\tfrac{2}{z}-1\right), \text{ and} \tag{8.34}$$

$$p_{n,k}(z) = \Gamma(-\alpha)\left(\sum_{j=0}^{n+k} \frac{(2n+k-j)!(n+k)!(-n-\alpha)_j}{(2n+k)!j!(n+k-j)!} z^j - q_{n,k}(z)\right), \tag{8.35}$$

where $C = C(n,k)$ is a constant determined by the condition $q_{n,k}(0) = 1$.

(ii) The nodes of the Gaussian quadratures described in Theorems 39 and 40 are given by $x_i = (y_i+1)/2$, where $y_i \in (-1,1)$ are the roots of the Jacobi polynomials $P_n^{(\alpha,k-\alpha)}(y)$.

Proof. We recall the results of Section 2.2.4, specifically that a generalized tempered stable process defined by (8.33) has a Lévy density of the form

$$\pi(x) = \frac{e^{-x}}{x^{1+\alpha}}, \quad x > 0,$$

from which we may obtain the representing measure

$$\mu(\mathrm{d}u) = \mathbb{I}(1<u)\frac{(u-1)^\alpha}{\Gamma(1+\alpha)}\mathrm{d}u,$$

by inverting the Laplace transform. Therefore,

$$\mu(\mathrm{d}u) \approx \mathbb{I}(1<u)(u-1)^\alpha \mathrm{d}u, \quad \text{and} \quad v^2\mu^*(\mathrm{d}v) \approx \mathbb{I}(0<v<1)v^{-\alpha}(1-v)^\alpha \mathrm{d}v,$$

where by "$\approx$" we understand "equal, up to a multiplicative constant". This shows that the orthogonal polynomials with respect to the measure $v^{2+k}\mu^*(\mathrm{d}v)$ are given by the shifted Jacobi polynomials $P_n^{(\alpha,k-\alpha)}(2z-1)$. Formula (8.34) follows from this fact and Theorems 39 and 27; statement (ii) then follows from Theorem 26.

Formula (8.35) can be derived from a known formula for the Padé approximant of the

binomial function $(1-z)^\alpha$. According to the last equation in [59], this has the form

$$\sum_{j=0}^{n+k} \frac{(2n+k-j)!(n+k)!(-n-\alpha)_j}{(2n+k)!j!(n+k-j)!} z^j,$$

where $(\cdot)_j$ is the Pochhammer symbol (See Appendix A pg. 182). To get (8.35) we then employ the following fact: if $m \geq n \geq 1$ and $p(z)/q(z)$ is the $[m/n]$ Padé approximant to $f(z)$, then $a(p(z)-q(z))/q(z)$ is the $[m/n]$ approximant to $a(f(z)-1)$. This fact is easy to deduce from the definition of the Padé approximant. □

8.3.1 Approximating VG, CGMY and NIG processes

The results of Propositions 3 and 4 can be used to construct explicit approximations of the Laplace exponents of arbitrary VG, CGMY, and NIG processes. There are two methods for doing this: (i) we can construct a process with two-sided jumps as a difference of processes with only positive jumps or (ii) we can represent a two-sided process as a scaled Brownian motion with drift subordinated by a generalized tempered stable subordinator S – the idea is to replace S with a hyper-exponential approximation.

We begin with the first method. From our discussion in Section 2.2.4 and (2.25) we know that a VG process X without drift or Gaussian component is just the difference of two gamma subordinators. This means that X has a Laplace exponent of the form

$$\psi(z) = \psi_\rho(z) + \psi_{\hat{\rho}}(-z),$$

where $\psi_\rho(z) = -c\log(1-z/\rho)$ is the Laplace exponent of a gamma subordinator X^ρ with parameters $c, \rho > 0$ and $\psi_{\hat{\rho}}(z) = -c\log(1-z/\hat{\rho})$ is the Laplace exponent of a gamma subordinator $X^{\hat{\rho}}$ with parameters $c, \hat{\rho} > 0$. Using Proposition 3 we can easily find explicit formulas for $\psi_\rho^{[n+k/n]}(z)$ and $\psi_{\hat{\rho}}^{[n+k/n]}(z)$, and from Theorem 39 we know these functions are the Laplace exponents of hyper-exponential subordinators which, by definition of the Padé approximant, match the first $2n+k$ moments of the processes X^ρ and $X^{\hat{\rho}}$ respectively. Therefore, the function $\psi_\rho^{[n+k/n]}(z) + \psi_{\hat{\rho}}^{[n+k/n]}(-z)$ should be a good approximation of $\psi(z)$, and the associated process, which is the Laplace exponent of the difference of two hyper-exponential subordinators, should be a good approximation of X.

For a CGMY process X we consult (2.27) to see that we may use exactly the same ap-

proach. That is, the Laplace exponent of X (with $\mu = 0$) has the form

$$\psi(z) = \psi_M(z) + \psi_G(-z),$$

where $\psi_M(z) = \Gamma(-Y)C((M-z)^Y - M^Y)$ is the Laplace exponent of a generalized tempered stable process with only positive jumps and parameters C, $M > 0$, and $Y \in (0,1) \cup (1,2)$; $\psi_G(z)$ is identical with $M = G$. Now we use Proposition 4 to construct an explicit formula for the approximation $\psi_M^{[n+k/n]}(z) + \psi_G^{[n+k/n]}(-z)$. Theorem 40 tells us the associated hyper-exponential process should be a good approximation of X.

The second procedure for obtaining explicit approximations uses the technique of subordination which was discussed briefly in Section 2.2.4. From there we know that any VG, CGMY, or NIG process can be obtained by replacing the deterministic time scale of a scaled Brownian motion with drift by a generalized tempered stable subordinator. Of course, we can also subordinate a scaled Brownian motion with drift by a hyper-exponential process. The following proposition shows us that the result of this procedure is again a hyper-exponential process.

Proposition 5. *Assume that Y is a hyper-exponential subordinator and W is an independent Brownian motion. Then for all $\sigma > 0$ and $a \in \mathbb{R}$ the process $Z_t := \sigma W_{Y_t} + aY_t$ is also a hyper-exponential process.*

Proof. Denote the Laplace exponent of Y as $\psi_Y(z)$. Since Y is a hyper-exponential process, $\psi_Y(z)$ is a rational function. We know from Section 2.2.4 that the Laplace exponent of Z is given by $\psi_Z(z) = \psi_Y(\sigma^2 z^2/2 + az)$, therefore it is also a rational function. Proposition 2.1 in [60] tells us that the process Z is a completely monotone Lévy process. This fact and the rationality of $\psi_Z(z)$ prove (by application of Theorem 23 and Lemma 2) that Z is also a hyper-exponential process. □

As discussed above, the VG process can be obtained as a scaled Brownian motion with drift, subordinated by a Gamma process Y. Proposition 3 gives us an explicit rational approximation of the Laplace exponent $\psi_Y(z)$, therefore, from Proposition 5 we obtain an explicit rational approximation $\tilde{\psi}(z)$ of the Laplace exponent of the original VG process. From Proposition 5 we also know that $\tilde{\psi}(x)$ is the Laplace exponent of a hyper-exponential process.

The same ideas can be applied to the NIG process, which we recall is defined as a scaled

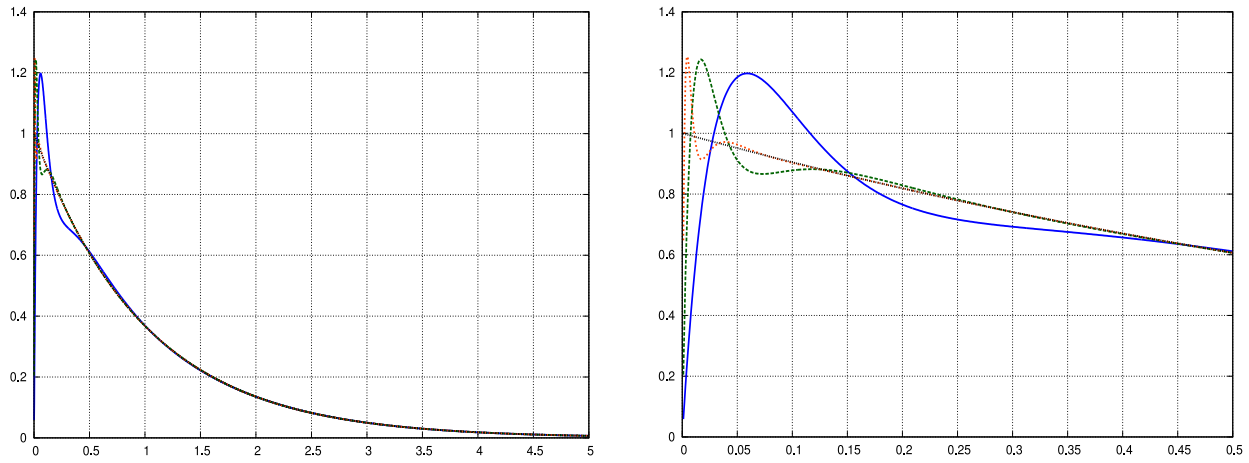
Brownian motion with drift subordinated by an inverse Gaussian subordinator Y. The process Y has a Laplace exponent of the form $\psi_Y(z) = (1 - \sqrt{1 - \kappa z})/\kappa$. Proposition 4 gives us an explicit rational approximation of $\psi_Y(z)$, and therefore from Proposition 5 we obtain an explicit rational approximation $\tilde{\psi}(z)$ of the Laplace exponent of the original NIG process. By invoking Proposition 5 once again we may identify $\tilde{\psi}(z)$ as the Laplace exponent of a hyper-exponential process.

Both of the approximation methods described above are desirable for a number of reasons. First, we get explicit formulas, which is preferable to calculating coefficients of Padé approximants by solving systems of linear equations. Second, if we wish to determine the parameters of the approximating hyper-exponential processes, our task is simplified since the nodes of the Gaussian quadrature are the roots of the Jacobi polynomials. In many cases, finding the roots will require no calculation because there exist extensive tables with this information. Even if calculation is required, finding the roots can be achieved by one of several known, efficient numerical algorithms (see for example [48] or Section 3.6 in [108]). An additional benefit of the first method, is that we are free to choose the degree of the Padé approximant for each one-sided process separately. This may be helpful in applications. For example, we may want to approximate negative jumps more accurately than positive jumps when pricing down-and-out barrier options.

We note that neither of the two methods gives an optimal approximation in the moment matching sense of Property (iii) in Theorem 37. To see this, suppose $X \in \mathcal{CM}(\hat{\rho}, \rho)$ with Laplace exponent $\psi(z)$. Further, suppose that we use *any* method other than that of Theorem 37 to generate an approximating hyper-exponential process $\tilde{X}$ whose Laplace exponent $\tilde{\psi}(z)$ has a numerator of degree $n+1$ and a denominator of degree n. By definition of $\psi^{[n+1/n]}(z)$, the corresponding two-sided approximating process of Theorem 37 must match at least as many moments of X as $\tilde{X}$. We perform a numerical comparison of the one-sided and two-sided approach in Section 8.4.

8.3.2 Extensions of the approximation algorithm

There are two ways in which Theorems 37, 39 and 40 can be generalized. First, there is an almost trivial (but potentially useful) generalization where we consider Padé approximants not at 0 but at another point $a \in (-\hat{\rho}, \rho)$. Then the statements of Theorems 37, 38, 39 and 40 would be true, provided that we replace the Padé approximant $\psi^{[n+k/n]}(z)$ (centered at 0) by $\psi^{[n+k/n]}(z) - \psi^{[n+k/n]}(a)$ (centered at a). This fact can be readily established using the



(a) (b)

Figure 8.1: The graph of $x\pi(x)$ (black curve) and $x\pi^{[n/n]}(x)$, where $\pi(x) = \exp(-x)/x$ is the Lévy density of the Gamma subordinator, and $\pi^{[n/n]}(x)$ is given by (8.23). Blue, green and red curves correspond n equal to 5, 10, and 20 respectively.

Escher transform (see pg. 78 in [76]), which maps a Lévy process $X \in \mathcal{CM}(\hat{\rho}, \rho)$ defined by Laplace exponent $\psi(z)$ into a process $\tilde{X} \in \mathcal{CM}(\hat{\rho} + a, \rho - a)$, defined by Laplace exponent $\tilde{\psi}(z) = \psi(a + z) - \psi(a)$.

The second generalization is to use the technique outlined in Chapter 5 involving Cauchy interpolants and Pick functions. That is, for $X \in \mathcal{CM}(\hat{\rho}, \rho)$ with Laplace exponent $\psi(z)$ we solve a Cauchy interpolation problem for the Pick function $z^{-1}\psi(z)$. From Theorems 23 and 24, Lemma 2, and the discussion on pg. 98 we know that for odd N this problem always has a unique solution $\psi_N(z)$ such that $z\psi_N(z)$ is the Laplace exponent of a hyper-exponential process. We recall that by definition $\psi_N(z)$ is a rational function with $\deg(\psi_N) < N/2$ and

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j}\psi_N(z)\bigg|_{z=z_i} = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\frac{\psi(z)}{z}\bigg|_{z=z_i}, \quad 1 \le i \le k,\ 0 \le j \le \beta_i,$$

where $N = \sum_{1\le i\le k}(\beta_i + 1)$, and $z_i \in (-\hat{\rho}, \rho)$, $1 \le i \le k$.

8.4 Numerical results

In this section we discuss a number of numerical experiments, which demonstrate the efficiency and effectiveness of the approximations. As a first example, we consider the Gamma process X defined by the Laplace exponent $\psi(z) = -\ln(1 - z)$. We compute the Lévy

density $\pi_n(x)$ corresponding to the approximation $\psi^{[n/n]}(z)$; the latter is given explicitly in Proposition 3. The Lévy density of X is $\pi(x) = \exp(-x)/x$, therefore in order to avoid the singularity at $x = 0$ we compare $x\pi(x) \equiv \exp(-x)$ and $x\pi_n(x)$. The results are presented in Figure 8.1. We see that even with the small value of $n = 5$ the tail of $\pi_n(x)$ matches the tail of $\pi(x)$ very well, and as n increases the approximation converges very rapidly (as long as x is not too close to zero).

Next, we compare the CDFs of the random variables X_t, $t \in \{1, 2\}$ with the CDFs of their approximations $X_t^{(n,k)}$, which are defined by the Laplace exponents $\psi^{[n+k/n]}(z)$, $k \in \{0, 1, 2\}$ (see Proposition 3). While the CDFs for the Gamma process are known explicitly

$$\mathbb{P}(X_1 \le x) = 1 - e^{-x}, \quad \text{and} \quad \mathbb{P}(X_2 \le x) = 1 - (x+1)e^{-x},$$

to compute CDFs for the processes $X^{(n,k)}$ we proceed numerically. We define the numbers r_i as the coefficients in the asymptotic expansion

$$\psi^{[n+k/n]}(z) = r_2 z^2 + r_1 z + r_0 + O(1/z), \quad z \to \infty,$$

and further define

$$\phi_{n,k}(z) := \begin{cases} e^{t\psi^{[n/n]}(z)} - e^{tr_0}, & \text{if } k = 0, \\ e^{t\psi^{[n+1/n]}(z)} - e^{tr_0 + tr_1 z}, & \text{if } k = 1, \\ e^{t\psi^{[n+2/n]}(z)}, & \text{if } k = 2. \end{cases}$$

The CDFs for the approximating processes are computed by numerical Fourier inversion

$$\mathbb{P}(X_t^{(n,k)} \le x) = 1 - \frac{e^{-cx}}{\pi} \mathrm{Re}\left[\int_{\mathbb{R}+} \phi_{n,k}(c+iu) e^{-iux} \frac{\mathrm{d}u}{c+iu}\right] - e^{tr_0} \mathbb{I}(x \le r_1 t)\delta_{k,1}, \qquad (8.36)$$

where $x > 0$ and $c \in (0, 1)$.

Let us explain the intuition behind the formula corresponding to $k = 0$; the other cases can be treated similarly. According to Theorem 39, the process $X^{(n,0)}$ is a compound Poisson hyper-exponential process with intensity $-r_0$ (i.e. $\lambda = -r_0$ in (2.10)), thus its distribution has an atom at zero: $\mathbb{P}(X_t^{(n,0)} = 0) = \exp(tr_0)$. If we subtract the atom at zero, we obtain

$\epsilon_{n,k}(1)$	$k=0$	$k=1$	$k=2$
$n=5$	$1.1e-2$	$1.1e-2$	$8.8e-3$
$n=10$	$2.8e-3$	$3.4e-3$	$2.8e-3$
$n=15$	$1.3e-3$	$1.6e-3$	$1.4e-3$
$n=20$	$7.5e-4$	$9.3e-4$	$8.1e-4$

$\epsilon_{n,k}(2)$	$k=0$	$k=1$	$k=2$
$n=5$	$3.3e-4$	$3.2e-4$	$5.4e-4$
$n=10$	$2.6e-5$	$2.8e-5$	$5.6e-5$
$n=15$	$5.4e-6$	$6.4e-6$	$1.3e-5$
$n=20$	$1.8e-6$	$2.1e-6$	$4.6e-6$

Table 8.1: The values of $\epsilon_{n,k}(t) := \max_{x\geq 0} |\mathbb{P}(X_t \leq x) - \mathbb{P}(X_t^{(n,k)} \leq x)|$, where X is the Gamma process with $\psi(z) = -\ln(1-z)$ and the process $X^{(n,k)}$ has Laplace exponent $\psi^{[n+k/n]}(z)$.

an absolutely continuous measure

$$\nu_t(\mathrm{d}x) := \mathbb{P}(X_t^{(n,0)} \in \mathrm{d}x) - e^{tr_0}\delta_0(\mathrm{d}x), \tag{8.37}$$

which has Laplace transform

$$\int_{\mathbb{R}} e^{xz}\nu_t(\mathrm{d}x) = e^{t\psi^{[n/n]}(z)} - e^{tr_0} = \phi_{n,0}(z), \quad \mathrm{Re}(z) < 1.$$

Since $\nu_t(\mathrm{d}x)$ is absolutely continuous with total mass $1 - \exp(tr_0)$, we can find the CDF corresponding to this measure by the inverse Laplace transform

$$\nu_t((0,x)) = 1 - e^{tr_0} - \frac{e^{-cx}}{\pi}\mathrm{Re}\left[\int_{\mathbb{R}^+} \phi_{n,0}(c+iu)e^{-iux}\frac{\mathrm{d}u}{c+iu}\right], \tag{8.38}$$

(see also the discussion in Section 5.2.1). Note that the integral in (8.38) converges absolutely, since $\phi_{n,0}(c+iu) = O(1/u)$ as $u \to \infty$. Formula (8.36) now follows directly from (8.37) and (8.38).

We may numerically evaluate (8.36) via a change of contour, followed by an application of Filon's method. We use 10^6 discretization points on a domain of numerical integration of $[0, 500]$ in the application of Filon's method. The maximum errors calculated over 100 points in the interval $(0, 10)$ are presented in Table 8.1. We see that the CDF of $X_t^{(n,k)}$ converges to X_t, and the convergence seems to be faster for $t = 2$ than it is for $t = 1$.

The remaining examples relate to pricing European, Asian, and barrier options. We will

work with the following two processes: the VG process V defined by the Laplace exponent

$$\psi_V(z) = \mu z - c\log\left(1 + \frac{z}{\hat{\rho}}\right) - c\log\left(1 - \frac{z}{\rho}\right),$$

and parameters

$$(\hat{\rho}, \rho, c) = (56.4414,\ 21.8735,\ 5.0),$$

and the CGMY process Z defined by the Laplace exponent

$$\psi_Z(z) = \mu z + C\Gamma(-Y)\left[(M - z)^Y - M^Y + (G + z)^Y - G^Y\right],$$

and parameters

$$(C, G, M, Y) = (1,\ 8.8,\ 14.5,\ 1.2).$$

Note that V is a process with infinite activity and finite variation, whereas Z is an infinite variation process. Both of these processes have zero Gaussian component. The process V is considered by the authors of [64]; later we will use their numerical results as a benchmark for our computations.

Our approach from here on is to compare a benchmark option price (for a variety of options) with a price calculated using one of four possible approximations. The first approximation is based on the $[n+1/n]$ Padé approximant for the process with two-sided jumps from Theorem 37. The other three approximations are based on the algorithm presented in Section 8.3.1, which considers the process as a difference of two processes having only positive jumps, and uses the explicit $[N + k/N]$ Padé approximants from Propositions 3 and 4. Note that the first approximation will result in a rational function of degree $n + 1$, while the other three approximations result in a rational function of degree $2N + k$. In instances where we calculate multiple approximations, we set $n = 2N$ in order to make a fair comparison.

As usual, we define the stock price process as $A_t = A_0 \exp(X_t)$ (where $X \equiv V$ in the VG case or $X \equiv Z$ in the CGMY case). Further, we choose the value of μ so that the process $A_t \exp(-rt)$ is a martingale, i.e. we are working with a risk-neutral measure. To avoid confusion between different processes we will write $\psi_X(z)$ to denote the Laplace exponent of the process X for the remainder of this section.

	two-sided $[2N+1/2N]$	one-sided $[N/N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-1.58e-2	9.12e-2	7.02e-3	-3.02e-2
$N=2$	1.66e-3	-6.16e-3	4.80e-3	-7.82e-4
$N=3$	6.20e-4	-1.28e-3	-4.32e-5	6.78e-4
$N=4$	1.25e-4	1.88e-4	-1.98e-4	9.81e-5
$N=5$	-7.19e-5	8.82e-5	-2.62e-5	-2.40e-5
$N=7$	4.34e-6	-8.48e-6	5.82e-6	-1.71e-6
$N=9$	-7.72e-8	3.31e-7	-6.99e-7	7.35e-7
$N=12$	4.85e-7	-1.81e-8	4.97e-8	-6.10e-8
$N=15$	-8.56e-8	-1.37e-9	-3.31e-9	6.06e-9

Table 8.2: The error in computing the price of the European call option for the VG V-model. The benchmark price is 2.5002779303.

	two-sided $[2N+1/2N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-2.75e-2	1.93e-2	-3.72e-3
$N=2$	-4.86e-6	-4.19e-6	9.5e-5
$N=3$	4.80e-7	-1.48e-5	-2.54e-7
$N=4$	2.9e-8	6.41e-7	-1.55e-7
$N=5$	1.14e-9	5.58e-9	6.95e-9

Table 8.3: The error in computing the price of the European call option for the CGMY Z-model. The benchmark price is 11.9207826467.

First, we compute the price of a European call option with $A_0 = 100$, strike price $K = 100$, maturity $T = 0.25$ and interest rate $r = 0.04$. It is well known that this entails calculating

$$E(A_0, K, T) := e^{-rT}\mathbb{E}[(A_T - K)^+],$$

which we will do using the approach of Carr and Madan [30]. Since this is very similar to our method for pricing Asian options we will describe it only briefly here. We begin by defining

$$f_t(k) := \mathbb{E}[(A_t - e^k)^+], \quad \text{and} \quad k := \log(K),$$

and observing that computing $f_t(k)$ is equivalent to computing $E(A_0, K, t)$. Our objective is to find an explicit expression for the Laplace transform of $f_t(k)$, which we define as $\phi_t(z) := \int_{\mathbb{R}} f_t(k)e^{zk}\mathrm{d}k$ for suitable values of z. From the following calculation, we see that if $\psi_X(z+1)$ is finite on some strip $0 < \mathrm{Re}(z) < \alpha$, then for $\mathrm{Re}(z) \in (0, \alpha)$ the function $\phi_t(z)$ is also finite and can be expressed explicitly in terms of $\psi_X(z)$ as follows:

$$\phi_t(z) = \int_{\mathbb{R}} f_t(k)e^{zk}\mathrm{d}k = \frac{\mathbb{E}[A_t^{z+1}]}{z(z+1)} = A_0^{z+1}\frac{e^{t\psi_X(z+1)}}{z(z+1)}.$$

It is easy to see that $\psi_V(z+1)$ is finite for $\mathrm{Re}(z) \in (0, \rho - 1) = (0, 20.8735)$ and $\psi_Z(z+1)$ is finite for $\mathrm{Re}(z) \in (0, M - 1) = (0, 13.5)$. Furthermore, Theorems 37, 39, and 40 tell us that $\psi_X(z)$ is finite for any approximating process X for at least the same values of z. Therefore, for c_0 lying in the appropriate interval, we may compute $f_t(k)$ via the formula

$$f_t(k) = \frac{e^{-c_0 k}}{2\pi}\int_{\mathbb{R}} \phi_t(c_0 + iu)e^{-iuk}\mathrm{d}u = \frac{e^{-c_0 k}}{\pi}\int_{\mathbb{R}^+} \phi_t(c_0 + iu)e^{-iuk}\mathrm{d}u. \tag{8.39}$$

The last equality on the right-hand side of (8.39) follows from the fact that the real part of $\phi_t(c_0 + iu)$ is even in u and the imaginary part is odd. This in turn follows from the fact that $f_t(k)$ is real valued.

The benchmark prices for the original VG process V and the CGMY process Z were computed multiple times by applying Filon's method to (8.39) and using different discretizations and truncations of the integral. The resulting prices seem to be correct to within ±1.0e-9. The prices based on the approximating processes are presented in Table 8.2 (approximations of V) and Table 8.3 (approximations of Z). These are calculated by applying Filon's method to (8.39) with 10^7 discretization points on the domain $[0, 2000]$. We see that all four approximations perform very well, and already for $N = 4$ we obtain acceptable accuracy

of around 1.0e-4. We notice that the three approximations based on the explicit one-sided formulas have remarkably good accuracy. As we discussed on page 164, these are not optimal in the sense that we can find rational Laplace exponents of lower degree which match more moments of the original processes. However, this non-optimality does not seem to play any role here. We conclude that the one-sided approximations are superior to the two-sided approximation, in the sense that they have very good accuracy *and* are derived using explicit formulas.

We also note that all four approximations seem to be doing a better job in the case of the CGMY process Z. The likely cause of this is that Z is an infinite variation process so that Z_t has smooth density; this is not the case for the process V.

Next, we compute the price of an arithmetic, continuously monitored, fixed strike Asian call option. That is, we calculate the following familiar quantity

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right], \tag{8.40}$$

where we set the parameters as follows:

$$A_0 = 100, \quad r = 0.03, \quad T = 1.$$

The parameter K is set to 90 for the VG process and 110 for the CGMY process. In order to compute Asian option prices we use the technique (and discretization/domain of numerical integration) of Section 6.4.2. Since, to the best of the author's knowledge, there are no results in the literature for pricing these kinds of options for either the VG or CGMY process (other than by Monte Carlo methods), we must use our own benchmark calculated using a significantly large N. By experimenting with different ways of discretizing the resulting integrals in the inverse Laplace and inverse Mellin transform (see Section 6.4.2), we arrive at a benchmark price of 11.18859 for the process V and 9.95930 for the process Z. These benchmark prices seem to be correct to within $\pm$1.0e-5. The results for each N are compared to the benchmark price, and the errors are gathered in Table 8.4 for the process V and in Table 8.5 for the process Z.

We observe again, that convergence to the benchmark price is very rapid and that there is little difference in the rate of convergence between the one-sided and two-sided approximations. We note that we achieve an acceptable error of $\pm$1.0e-4 with a rational approximation

	two-sided $[2N+1/2N]$	one-sided $[N/N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-1.87e-3	1.01e-3	-1.82e-3	9.88e-4
$N=2$	9.49e-5	2.89e-4	-6.33e-5	3.27e-5
$N=3$	1.30e-6	8.85e-6	-4.24e-6	3.99e-6
$N=4$	-2.83e-6	1.07e-6	-1.36e-6	3.16e-7
$N=5$	-1.11e-7	-2.48e-8	-5.91e-7	-3.81e-7

Table 8.4: The error in computing the price of the Asian option for the VG V-model. The benchmark price is 11.188589 (calculated using the [91/90] two-sided approximation).

of degree 5.

Our final example is related to pricing down-and-out barrier put options. That is, we wish to calculate

$$D(A_0,K,B,T) := e^{-rT}\mathbb{E}\left[(K-A_T)^+\mathbb{I}\left(\inf_{0\le t\le T} A_t > B\right)\right],$$

where B is the barrier level. We calculate barrier option prices for the process V, for four values $A_0 \in \{81, 91, 101, 111\}$ and $K = 100$, $B = 80$, $r = 0.04879$ and $T = 0.5$. We use the prices computed in [64] as the benchmark (these prices seem to be accurate to about $\pm$1.0e-3).

In order to compute the prices of down-and-out put options for hyper-exponential processes we use the method of Section 3.3. We recall that, due to the Wiener-Hopf factorization, we can obtain the price $D(A_0,K,B,T)$ by calculating the inverse Laplace transform of the function

$$q^{-1}F(q) := q^{-1}\mathbb{E}[(k-e^{S_{\mathbf{e}(q)}+I_{\mathbf{e}(q)}})^+\mathbb{I}(I_{\mathbf{e}(q)} > b)],$$

where $k = K/A_0$ and $b = \log(B/A_0)$. Since we know the distribution of the random variables $S_{\mathbf{e}(q)}$ and $I_{\mathbf{e}(q)}$ for hyper-exponential processes (see Section 3.2.1) we can derive explicit an

	two-sided $[2N+1/2N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	1.88e-4	7.42e-4	-1.19e-3
$N=2$	4.03e-6	9.05e-5	5.39e-6
$N=3$	-3.58e-7	-2.64e-6	7.93e-8
$N=4$	-3.88e-7	-1.01e-7	-1.21e-7
$N=5$	-5.26e-7	-2.47e-7	-2.49e-7

Table 8.5: The error in computing the price of the Asian option for the CGMY Z-model. The benchmark price is 9.959300 (calculated using the [91/90] two-sided approximation).

formula for $q^{-1}F(q)$ which can easily be inverted by numerical means.

We show here how to derive an explicit formula for the case of an in-the-money ($K > A_0$) option for a hyper-exponential process X defined by $(a, \sigma^2, \pi)_{h\equiv 0}$ with $\sigma^2 > 0$. Derivation of the formula for other cases is similar. First, we recall that $I_{\mathbf{e}(q)}$ and $S_{\mathbf{e}(q)}$ are independent and have known densities

$$p_I(y) = \sum_{i=1}^{\hat{N}+1} \hat{\beta}_i \hat{\zeta}_i e^{\hat{\zeta}_i y}, \quad y < 0, \qquad \text{and} \qquad p_S(x) = \sum_{j=1}^{N+1} \beta_j \zeta_j e^{-\zeta_j x}, \quad x > 0,$$

respectively. The coefficients $\{\hat{\beta}_i\}_{1\le i\le \hat{N}+1}$ and $\{\beta_j\}_{1\le j\le N+1}$ have the form

$$\hat{\beta}_i := \prod_{k\neq i} \frac{1-\frac{\hat{\zeta}_i}{\hat{\rho}_k}}{1-\frac{\hat{\zeta}_i}{\hat{\zeta}_k}}, \qquad \text{and} \qquad \beta_j := \prod_{k\neq j} \frac{1-\frac{\zeta_j}{\rho_k}}{1-\frac{\zeta_j}{\zeta_k}},$$

where, as usual $\{-\hat{\zeta}_i\}_{1\le i\le \hat{N}+1}$ and $\{\zeta_j\}_{1\le j\le N+1}$ denote the solutions of $\psi_X(z) = q$ and $\{-\hat{\rho}_i\}_{1\le i\le \hat{N}}$ (resp. $\{\rho_j\}_{1\le j\le N}$) are the negative (resp. positive) poles of $\psi_X(z)$. This means

	$S_0 = 81$	$S_0 = 91$	$S_0 = 101$	$S_0 = 111$
Benchmark	3.39880	7.38668	1.40351	0.04280
$N = 2$	3.44551	7.39225	1.40527	0.04233
$N = 4$	3.40209	7.38957	1.40329	0.04258
$N = 6$	3.39910	7.38939	1.40332	0.04258
$N = 8$	3.39856	7.38936	1.40332	0.04258
$N = 10$	3.39853	7.38936	1.40332	0.04258

Table 8.6: Barrier Option prices calculated for the VG process V-model. Benchmark prices obtained from [64], Table 4, Column 2.

we that may write

$$\begin{aligned} F(q) &= \int_{\mathbb{R}^-}\int_{\mathbb{R}^+} (k - e^{x+y})^+ \mathbb{I}(y > b) p_S(x) p_I(y) \mathrm{d}x\mathrm{d}y \\ &= \int_0^{-b}\int_0^{\log(k)+y} (k - e^{x-y}) p_S(x) p_I(-y) \mathrm{d}x\mathrm{d}y \\ &= \sum_{i=1}^{\hat{N}+1}\sum_{j=1}^{N+1} \frac{\hat{\beta}_i \beta_j}{\zeta_j - 1} \left(k(e^{b\hat{\zeta}_i} - 1)(1 - \zeta_j) - \frac{k^{1-\zeta_j}\hat{\zeta}_i(e^{b(\zeta_j+\hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + \zeta_j} + \frac{\hat{\zeta}_i\zeta_j(e^{b(1+\hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + 1} \right). \end{aligned}$$

To calculate prices we invert $q^{-1}F(q)$ by our standard numerical procedure for oscillatory integrals, namely Filon's method. To implement this we use a domain of numerical integration $[0, 4000]$ and $80,000$ discretization points. In this case we calculate prices only for the one-sided $[N + 1/N]$ approximations, which are presented in Table 8.6. We see that in almost all cases the convergence is very rapid, and we are able to match the first four digits of the benchmark price. The convergence is somewhat slower for $S_0 = 81$, which is to be expected: it is known that the behavior of the price near the barrier is very sensitive to the nature of the small jumps of the underlying process (see [20]). Therefore, we may expect that our results will not be very precise when S_0 is close to B, since we are approximating a process with jumps of infinite variation by a compound Poisson process with drift.

8.5 Comparison with existing techniques

As mentioned in the Introduction, there are other methods for approximating processes with completely monotone jumps by hyper-exponential processes. The first of these was proposed by Jeannin and Pistorius in [60], and the second one by Crosby, Le Saux and Mijatović in [35]. The current work was initially inspired by these two papers, therefore it is a valuable exercise to compare the existing results with the new results presented here.

The approach of Jeannin and Pistorius is based on minimizing the L_2 distance between the target Lévy density $\pi(x)$ and the approximating hyper-exponential Lévy density $\pi_n(x)$. More precisely, we seek a hyper-exponential Lévy density $\pi_n(x)$ of the form (8.6) which minimizes

$$\Delta_{n,\epsilon} = \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} (\pi(x) - \pi_n(x))^2 \mathrm{d}x, \tag{8.41}$$

where $\pi(x)$ is the target Lévy density of a process with completely monotone jumps. Note that removing an ϵ-neighborhood of zero in the domain of integration in (8.41) *is necessary*, because otherwise the integral may not converge. According to the definition of $\pi_n(x)$ in (8.6), the quantity $\Delta_{n,\epsilon}$ can be considered as a function of $n = 2N + 2\hat{N}$ parameters $\{a_i, \rho_i\}_{1\le i\le N}$ and $\{\hat{a}_i, \hat{\rho}_i\}_{1\le i\le \hat{N}}$, and ideally one would try to find the absolute minimum of this function in order to get the best fit of $\pi_n(x)$ to $\pi(x)$. However, this optimal approach results in a complicated non-linear minimization problem, and it is much easier to fix the parameters $\{\rho_i\}_{1\le i\le N}$ and $\{\hat{\rho}\}_{1\le i\le \hat{N}}$ (which specify the exponents of the exponential functions in (8.6)) and to minimize over the remaining parameters $\{a_i\}_{1\le i\le N}$ and $\{\hat{a}_i\}_{1\le i\le \hat{N}}$. This simplification results in a linear problem, which can be easily solved numerically.

Next, let us summarize the main ideas behind the method of Crosby, Le Saux and Mijatović [35]. We start with a Lévy process with completely monotone jumps and zero Gaussian component. We use (8.13), choose a parameter $A > 0$ large enough, and derive the following approximation

$$\begin{aligned}
\psi(z) &= az + z^2 \int_{\mathbb{R}} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} = az + z^2 \int_{\mathbb{R}\setminus[-A,A]} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} + z^2 \int_{[-A,A]} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} \\
&= az + z^2 \int_{\mathbb{R}\setminus[-A,A]} \frac{1}{1 - z/u} \frac{\mu(\mathrm{d}u)}{|u|^3} + z^2 \int_{[-A,A]} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} \\
&\approx az + z^2 \int_{\mathbb{R}\setminus[-A,A]} \frac{\mu(\mathrm{d}u)}{|u|^3} + z^2 \int_{[-A,A]} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} =: \tilde{\psi}(z),
\end{aligned}$$

where in the last step we have used the fact that $|u| > A \gg 1$ and therefore $1 - z/u$ can be approximated by 1. The above approximation is the first step in the method of Crosby et al., and it gives us the Laplace exponent of a Lévy process $\tilde{X}$ with a small (but non-zero) Gaussian component

$$\sigma^2 = 2 \int_{\mathbb{R}\setminus[-A,A]} \frac{\mu(\mathrm{d}u)}{|u|^3}.$$

The process $\tilde{X}$ has Lévy measure $\tilde{\pi}(x)$, given by (8.2) with $\mu(\mathrm{d}x)$ replaced by $\mathbb{I}(|x| \leq A)\mu(\mathrm{d}x)$. It is easy to see that $\tilde{\pi}(x)$ is a finite measure, and therefore $\tilde{X}$ has compound Poisson jumps. Intuitively, the effect of this first step is to replace the jumps of X (which could be of infinite activity or infinite variation) by compound Poisson jumps and a small Gaussian component. The second step in the method of Crosby et al. is to discretize the integral

$$\int_{[-A,A]} \frac{\mathrm{sign}(u)}{u - z} \frac{\mu(\mathrm{d}u)}{u^2} \approx \sum \frac{\mathrm{sign}(x_i)}{x_i - z} \frac{w_i}{x_i^2}$$

via the Gauss-Legendre quadrature (a Gaussian quadrature on the interval $[-A, A]$ with respect to the Lebesgue measure). Combining these two steps results in a Laplace exponent of an approximating hyper-exponential process.

The method we have discussed in this chapter is quite similar to the approach of Crosby, Le Saux and Mijatović. Instead of their first approximating step we perform a change of variables $u \mapsto 1/v$ in the integral (8.13). This simple trick and the assumption that the Lévy measure has exponential tails give us a finite domain of integration in the v-variable in (8.14), so that we can apply the Gaussian quadrature with respect to the measure $|v|^3\mu^*(\mathrm{d}v)$. It turns out that this seemingly small modification has profound consequences. First, we do not need to truncate the integrals, and second, we do not require any external parameters (such as ϵ or A in the above two methods). Third, our approximating Laplace exponents $\psi_n(z)$ have a simple analytic interpretation as Padé approximants of $\psi(z)$, which allows us to borrow tools and ideas from the well-developed theory of rational approximations and orthogonal polynomials. Fourth, our approximation turns out to be optimal in the sense that the hyper-exponential process $X^{(n)}$ constructed in Theorem 36 matches $2n + 1$ moments of the target process X (see the statement of Theorem 36 (iii)). Note that this is the best that one can hope for: according to (8.10) the process $X^{(n)}$ has $2n + 1$ free parameters, therefore we can match at most $2n + 1$ moments of X. Finally, we show in Theorem 38 that our approximations converge exponentially in n, where n is the number of terms in the Lévy density; this fast convergence carries through to our calculations of option prices.

Appendices

Appendix A

Special Functions

The Gamma function $\Gamma(z)$

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x, \quad \mathrm{Re}(z) > 0.$$

Useful formulas & identities:

- Recurrence formula (Formula 6.1.15 in [2])

$$\Gamma(z+1) = z\Gamma(z)$$

- Reflection formula (Formula 6.1.17 in [2])

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}.$$

- Stirling's formula (Formula 6.1.37 in [2])

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^2} - \frac{571}{2488320z^4} + \cdots\right),$$
$$z \to \infty, \, |\arg(z)| < \pi.$$

- Limit formula (Formula 8.328.1 in [49])

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}.$$

- Euler's infinite product formula (Formula 6.1.3 in [2])

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n\geq 1} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where γ is the Euler-Mascheroni constant.

The Beta function $B(w, z)$

$$B(w, z) := \int_0^1 t^{w-1}(1-t)^{z-1}\mathrm{d}t, \quad \mathrm{Re}(w),\, \mathrm{Re}(z) > 0.$$

Useful formulas & identities

- (Formula 6.2.2 in [2])

$$B(w, z) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z, w).$$

The Digamma function $\psi(z)$

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Useful formulas & identities

- (Formula 8.361.8 in [49])

$$\psi(z) = \log(z) + \int_0^\infty e^{-tz} \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) \mathrm{d}t, \quad \mathrm{Re}(z) > 0.$$

- (Formula 6.3.2 in [2])

$$\psi(1) = -\gamma, \quad \text{and} \quad \psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma, \quad n \in \mathbb{N},\, n \geq 2,$$

where γ is the Euler-Mascheroni constant.

NOTE: We use the same symbol to denote the Laplace exponent of a Lévy process. From the context it should be clear which function we mean.

The Theta function $\theta_3(q)$ and the function $\Theta_k(x)$

$$\theta_3(q) := \begin{cases} 1 & q = 0 \\ \sum_{n=-\infty}^{\infty} q^{n^2} & q \neq 0 \end{cases}, \quad |q| < 1.$$

NOTE: Theta functions are properly functions of two variables, e.g. $\theta_3(z, q)$. The above definition results from setting z equal to 0. Since we do not require the variable z we omit it here; the interested reader is referred to Chapter 2 in [19].

$$\Theta_k(x) := \frac{\mathrm{d}^k}{\mathrm{d}x^k}\theta_3(e^{-x}) = \delta_{k,0} + 2\sum_{n\geq 1} n^{2k} e^{-n^2 x}, \quad k,\, x > 0.$$

The Dilogarithm function $\mathbf{Li}_2(z)$

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-t)}{t}\mathrm{d}t, \quad z \in \mathbb{C}.$$

Useful formulas & identities

- (Formula 4.1 in [82])

$$\mathrm{Li}_2(e^{i\theta}) = \sum_{n\geq 1} \frac{\cos(n\theta)}{n^2} + i\sum_{n\geq 1} \frac{\sin(n\theta)}{n^2}.$$

The Clausen function $\mathbf{Cl}_2(\theta)$

$$\mathrm{Cl}_2(\theta) := -\int_0^\theta \log\left(2\sin\left(\frac{t}{2}\right)\right)\mathrm{d}t = \sum_{n\geq 1} \frac{\sin(n\theta)}{n^2}, \quad \theta \in \mathbb{R}.$$

The Double-Gamma function $G(z;\tau)$

The following excerpt is borrowed almost directly from [68]. This reference contains all necessary background information on the double-gamma function which is needed in order to apply and understand the theorems in this work. Original material on the double-gamma function can be found in [10, 11].

The original definition of Barnes [10] for the double-gamma function is

$$G(z;\tau) := \frac{z}{\tau} e^{az/\tau + bz^2/(2\tau)} \times \prod_{m\geq 0} \prod_{n\geq 0}{}' \left(1 + \frac{z}{m\tau + n}\right) e^{-z/(m\tau+n) + z^2/(2(m\tau+n)^2)}, \quad |\arg(\tau)| < \pi,\ z \in \mathbb{C},$$

where $\prod'$ indicates that the term corresponding to $m = 0$, $n = 0$ is omitted, and a and b are constants which are defined below. An alternative formulation, also due to Barnes, is that the double-gamma function can be expressed as a single infinite product of gamma functions as follows:

$$G(z;\tau) = \frac{1}{\tau\Gamma(z)} e^{\tilde{a}z/\tau + \tilde{b}z^2/(2\tau^2)} \prod_{m\geq 1} \frac{\Gamma(m\tau)}{\Gamma(z+m\tau)} e^{z\psi(m\tau) + z^2/2\psi'(m\tau)}.$$

In the above expression, $\psi(z)$ is the digamma function. The constants in the two formulas are related in the following manner:

$$\tilde{a} = a - \gamma\tau, \quad \text{and} \quad \tilde{b} = b + \frac{\pi^2\tau^2}{6},$$

where $\gamma = -\psi(1)$ is the Euler-Mascheroni constant. The constants $\tilde{a}$ and $\tilde{b}$ are given by

$$\begin{aligned} \tilde{a} &= \frac{\tau}{2}\log(2\pi\tau) + \frac{1}{2}\log(\tau) - \tau C(\tau), \text{ and} \\ \tilde{b} &= -\tau\log(\tau) - \tau^2 D(\tau), \end{aligned}$$

where $C(\tau)$ and $D(\tau)$ are transcendental functions which can be computed as the following limits as $m \to \infty$:

$$\begin{aligned} C(\tau) = &\sum_{k=1}^{m-1} \psi(k\tau) + \frac{1}{2}\psi(m\tau) - \frac{1}{\tau}\log\left(\frac{\Gamma(m\tau)}{\sqrt{2\pi}}\right) \\ &- \frac{\tau}{12}\psi'(m\tau) + \frac{\tau^3}{720}\psi^{(3)}(m\tau) + O(m^{-5}), \text{ and} \\ D(\tau) = &\sum_{k=1}^{m-1} \psi'(k\tau) + \frac{1}{2}\psi'(m\tau) - \frac{1}{\tau}\psi(m\tau) \\ &- \frac{\tau}{12}\psi''(m\tau) + \frac{\tau^3}{720}\psi^{(4)}(m\tau) + O(m^{-6}). \end{aligned}$$

The Modified Bessel function of the first kind $\mathcal{I}_\nu(z)$

$$\mathcal{I}_\nu(z) := \sum_{k\geq 0} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}, \quad v \in \mathbb{R},\, z \in \mathbb{C}.$$

NOTE: The standard notation for the modified Bessel function of the first kind is I_ν. We use $\mathcal{I}_\nu$ to avoid confusion with the notation for the exponential functional.

The Generalized Hypergeometric Series ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$

First we define the ***Pochhammer symbol*** for any complex number a as

$$(a)_n := \begin{cases} 1 & n = 0 \\ \prod_{k=0}^{n-1}(a+k) & n \geq 1 \end{cases}. \tag{A.1}$$

A generalized hypergeometric series, is a formal power series of the form

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) := \sum_{n\geq 0} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!},$$
$$p,\, q \in \{0\} \cup \mathbb{N}, a_1, \ldots, a_p,\, b_1, \ldots, b_q,\, z \in \mathbb{C}.$$

Using the ratio test, one may prove that the domain of convergence of the series is: $\mathbb{C}$ when $p \leq q$; $\{|z| < 1\}$ when $p = q+1$; and $\{0\}$ when $p \geq q+2$. In the second case, when we can extend ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ to a domain outside its radius of convergence via analytic continuation, we refer to the extended function using the same notation. In these cases the term ***generalized hypergeometric function*** is often used. There are many special cases; for example, the term ***confluent hypergeometric function*** is used for functions of the form ${}_1F_1(a;b;z)$.

Meijer's G-Function $\mathbf{G}_{p,q}^{m,n}\left(z \left| \begin{smallmatrix} a_1\ldots,a_p \\ b_1,\ldots,b_q \end{smallmatrix} \right.\right)$

$$\mathbf{G}_{p,q}^{m,n}\left(z \left| \begin{matrix} a_1 \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right.\right) := \frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \mathrm{d}s, \quad z \in \mathbb{C},$$

where $m, n, p, q \in \mathbb{N} \cup \{0\}$ such that $0 \leq m \leq q$, $0 \leq n \leq p$, the poles of $\Gamma(b_j - s)$ do not coincide with the poles of $\Gamma(1 - a_k + s)$ for any $j \in \{1 \ldots m\}$ and $k \in \{1 \ldots n\}$, and γ is one

of three possible contours in the complex plane. See Sections 9.302, 9.303, 9.304 in [49] for details, and for formulas in terms of generalized hypergeometric series. If we define vectors $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$ then we may use the alternate notation $\mathbf{G}_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix}\right)$.

List of symbols

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