THE OPTIMAL DIVIDEND PROBLEM FOR TWO FAMILIES OF MEROMORPHIC LÉVY PROCESSES

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Abstract

De Finetti's optimal dividend problem [9] has been studied steadily since it was proposed in 1957. It asks two related questions: Can one find a dividend policy that maximizes the expectation of discounted future dividends of an insurance company until ruin? If so, is it possible to derive a formula for the value of this expectation under the optimal strategy (the value function)? These questions can be seen as useful generalizations of the the ruin problem, which is concerned only with the probability of ruin.

In De Finetti's original formulation, the wealth process of the insurance company was described using a simplistic random walk model. Since then, more realistic continuous time processes, such as the Cramér-Lundberg process or the Brownian motion, have taken the random walk's place. In the last decade researchers have considered an important extension of the problem which restricts the rate of dividends in order to ensure positive probability of survival (the restricted problem).

Despite a wealth of theoretical results [2, 3, 6, 9, 11–13, 24, 26] no numerical studies have been conducted. The goal of this thesis is to develop a numerical scheme that uses a recent theorem due to Kyprianou et. al. [19] and two families of Lévy processes introduced by Kuznetsov [14, 15] to calculate the value function for the restricted problem. The impact of model uncertainty is also investigated.

Contents

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1 Introduction

The ruin problem, which examines the probability with which an insurance company will go bankrupt, has been studied extensively since it was introduced by Lundberg at the beginning of the twentieth century. Since then, it has given rise to a thriving field of research which includes many important generalizations of the original problem. One such generalization involves the study of so-called discounted penalty functions which provide more detailed information about the ruin event. Simultaneously, there is significant interest in the optimal dividend problem which considers the shareholder's point of view. Specifically, the object of study is the amount of dividends paid to shareholders before ruin.

Bruno De Finetti [9] first proposed the optimal dividend problem in 1957. This can be stated as two related questions: Can one find a dividend policy that maximizes the expectation of discounted future dividends of an insurance company? If so, is it possible to derive a formula for the value of this expectation under the optimal strategy (the value function)? While De Finetti and others [24, 26] answered these questions under the condition that the company's wealth is modelled by a discrete time and space stochastic process, more recent authors have considered these questions for continuous time processes, such as the Cramér-Lundberg process [3, 11, 13] and Brownian motion [2, 12]. Currently, the problem is being considered in the more general context of the spectrally negative Lévy process [19, 22, 23].

Researchers in financial mathematics and actuarial science are divided on the idea of generalization and mathematical abstraction. In mathematics, theories which work in general are normally preferable to those that work in limited scope. The strength of generality is that it reveals the "big picture" and leads to a unified treatment of related problems. However, for practitioners in actuarial science or financial mathematics, a more pressing aim is to find simple computable solutions to the problems of modelling random phenomena, assessing risk, and valuing assets. As such, a general theory is only useful if it provides more accurate results, without significantly increasing the difficulty of obtaining solutions.

Studying problems in actuarial science in the context of Lévy processes is an example of such a generalization. Often the primary models are based on Brownian motion or the Cramér-Lundberg process. Since both of these are just simple Lévy processes, applying the general theory of Lévy processes is a natural extension (see [4, 20, 25] for a suitable introduction) . This extension provides powerful tools and results which help to treat the problem in full generality. Unfortunately, this comes at the price of more complicated theory, while the practical benefits have not yet been clearly demonstrated.

An important modification of the optimal dividend problem, hereafter referred to as the restricted problem, has been considered in a number of papers [2, 13, 19]. The modification restricts the maximum rate at which dividends can be paid. That is, in considering potentially optimal strategies, only those that pay dividends at a rate less than or equal to some positive finite value are considered. The impetus for this modification is that the optimal strategy in the original problem causes the company to be ruined in finite time. In order to avoid this scenario, the restricted problem was proposed. It was found that an appropriate upper bound δ on the maximum rate of dividend payment ensures that the company survives with positive probability. A recent paper [19] proves that for a large class of spectrally negative Lévy process a simple barrier strategy is optimal. This strategy, called the refraction strategy, prescribes paying no dividends when the net wealth of the company is below an optimal barrier b^*, and paying dividends at the fixed maximal rate δ when net wealth exceeds b^*.

This thesis provides a comprehensive review of important results from the optimal dividend problem's fifty-four year history. Two detailed examples are provided demonstrating proofs of optimality. These proofs are rather difficult and technical, even in the case where the underlying wealth process is a simple random walk. In fact, the original proof in the random walk case [26] has several important omissions and inconsistencies which are clarified and corrected in the present work . In order to lend practical application to the largely theoretical results in the existing literature, a numerical analysis is performed to calculate the value function for the optimal strategy in the restricted case. This analysis relies on the recently proved theorem in [19], and on two newly introduced families of Lévy processes belonging to the class of Meromorphic processes (M-processes), called the beta and theta families [14, 15, 17, 18]. In particular, it demonstrates that it is possible to calculate the value function for processes with all path types, not only for those that behave like Brownian motion or the Cramér-Lundberg process. In order to study model uncertainty three different models are calibrated to the same data and their qualitative and quantitative behaviour is compared. Surprisingly, the findings show that although the strategies differ in terms of the optimal barrier b^*, the value of the company is similar in all three cases. In a related discussion, the time of ruin $\sigma^{\pi_{b^*}}$ under the optimal strategy is considered for the Cramér-Lundberg process. A Monte-Carlo estimate of the distribution of $\sigma^{\pi_{b^*}}$ shows that it changes rapidly when δ is large, while the opposite is true for the value of the company. This provides evidence to support the intuition that a more aggressive strategy will increase the likelihood of ruin, and simultaneously shows that a careful (not too large or small) choice of δ can result in almost maximal possible dividends while maintaining a reasonably large probability of survival.

The thesis is organized as follows: Section 2 provides a brief introduction to Lévy processes, demonstrates some fundamental concepts with simple examples, introduces the theory of scale functions for spectrally negative Lévy processes, and introduces the beta and theta families. This material is central to understanding the proofs presented in later sections. Section 3 provides a thorough introduction to the optimal dividend problem. Examples are given as a way to introduce and provide an intuitive background for the more general results for spectrally negative Lévy processes. Finally, Section 4 discusses numerical procedures and their results.

2 A Brief Introduction to Lévy Processes

2.1 Some Basic Facts

Background information on Lévy processes can be found in [20] or a number of other sources. The following facts, taken largely from [20], will suffice to describe the current work.

Definition 1. *A process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties.*

(i) *The paths of X are right continuous with left limits $\mathbb{P}$-almost surely.*

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.*

(iv) *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.*

In this document it is assumed that X has a one-dimensional state space equal to $\mathbb{R}$. One can choose to omit condition (ii) in favour of $\mathbb{P}(X_0 = x) = 1$ to obtain a Lévy processes whose paths do not begin at the origin. In this case, the notation $\mathbb{P}_x$ and $\mathbb{E}_x$ will be used to denote the appropriate conditional probabilities and expectations, while the special case $\mathbb{P}_0$ will be described using notation $\mathbb{P}$.

Much can be said about the paths of a Lévy process from the form of its characteristic exponent. From that standpoint, the following theorem is essential.

Theorem 1. *For a Lévy process X there always exists a triple (a, σ, ν) where $a \in \mathbb{R}$, $\sigma \geq 0$, and ν is a measure supported in $\mathbb{R}\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge x^2)\nu(dx) < \infty \tag{2.1}$$

such that, for all $z \in \mathbb{R}$

$$\mathbb{E}[e^{izX_t}] = e^{-\Psi(z)t}, \tag{2.2}$$

where

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{izx} + izx\mathbf{1}_{\{|x|<1\}})\nu(dx), \quad z \in \mathbb{R}. \tag{2.3}$$

Conversely, the existence of a triple (a, σ, ν) satisfying $a \in \mathbb{R}$, $\sigma \geq 0$, and (2.1) implies the existence of a Lévy process satisfying (2.2) and (2.3).

The function $\Psi(z)$ is known as the characteristic exponent of the process X. If there exist $\alpha < 0 < \beta$ such that

$$\int_{|x|\geq 1} \mathrm{e}^{sx}\nu(dx) < \infty, \tag{2.4}$$

for all $\alpha < s < \beta$, then Ψ can be extended analytically to the strip $-\beta < \mathrm{Im}(z) < -\alpha$ for $z \in \mathbb{C}$. The Laplace exponent ψ, defined as

$$\psi(z) = -\Psi(-iz) = -az + 1/2\sigma^2 z^2 + \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{zx} - 1 - zx\mathbf{1}_{\{|x|<1\}})\nu(dx), \quad z \in \mathbb{C} \tag{2.5}$$

then exists on the strip $\alpha < \mathrm{Re}(z) < \beta$.

The Laplace exponent possesses some useful properties that give insight into the characteristics of X. Notice, that one can calculate ψ as follows

$$\psi(z) = \frac{1}{t} \log \mathbb{E}\left[\mathrm{e}^{zX_t}\right].$$

Rearranging gives the equality

$$\mathrm{e}^{t\psi(z)} = \mathbb{E}\left[\mathrm{e}^{zX_t}\right]. \tag{2.6}$$

One can differentiate both sides of (2.6), let $z \to 0$, and $t = 1$ to show that $\psi'(0) = \mathbb{E}[X_1] \in [-\infty, \infty)$. As a result the quantity $\psi'(0)$ determines the long term behaviour of the process. That is, when $\pm\psi'(0) > 0$, $\lim_{t\to\infty} X_t = \pm\infty$ almost surely so that the process drifts to $\pm\infty$. Otherwise, when $\psi'(0) = 0$, $\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty$ almost surely which shows that the process oscillates.

The measure ν is often simply called the Lévy measure or jump measure. Processes whose Lévy measures have the property $\nu(\mathbb{R}\backslash\{0\}) = \infty$ are said to have paths of *infinite activity*. This term is related to the number of "small" jumps paths of the process experience in any given interval of time. As might be expected from the the terminology, processes whose paths exhibit infinite activity experience infinitely many small jumps in any finite period. A Lévy process has paths of *bounded variation* if and only if its jump measure satisfies $\int_{\mathbb{R}\backslash\{0\}}(1 \wedge |x|)\nu(dx) < \infty$ and $\sigma = 0$. Here variation refers to the total variation of the paths of the process. Finally, an important concept for this thesis is that of the spectrally negative Lévy process.

Definition 2. *A spectrally negative Lévy process is one whose Lévy measure satisfies* $\nu(0, \infty) = 0$ *but whose paths are not monotone.*

Note that the first condition in this definition guarantees that the process experiences only negative jumps.

2.2 Examples

There are a number of classic processes that illustrate the concepts of the previous section. In each example $X = \{X_t : t \geq 0\}$ will denote the process in question. The value $x \geq 0$ will represent the initial value of the process, i.e. $X_0 = x$. If X is a model for the wealth of an insurance company, x can be regarded as the company's starting capital.

Brownian Motion

A process, used often in financial applications, is the scaled Brownian motion with drift. This has the form

$$X_t = x + \sigma W_t + ct. \tag{2.7}$$

In (2.7) $W = \{W_t : t \geq 0\}$ is a standard Brownian motion, $\sigma > 0$, and $a \in \mathbb{R}\backslash\{0\}$. From the definition of Brownian motion it is not difficult to show that X satisfies the conditions of Definition

1 and is therefore a Lévy process. Then, one can calculate

$$\mathbb{E}\left[\mathrm{e}^{izX_t}\right] = \mathrm{e}^{izct}\int_{-\infty}^{\infty}\mathrm{e}^{izx}\frac{1}{\sqrt{2\pi\sigma^2 t}}\mathrm{e}^{-x^2/2\sigma^2 t}dx = \mathrm{e}^{t(izc - z^2\sigma^2/2)} = \mathrm{e}^{-t(-izc + z^2\sigma^2/2)},$$

which shows that

$$\Psi(z) = -izc + \frac{1}{2}z^2\sigma^2.$$

Since ν is identically 0 one can identify X as a spectrally negative process, and from the properties of Brownian motion it is clear that X has paths of infinite variation. A theorem known as the Lévy-Itô decomposition (see [20] pg. 29) shows that every Lévy process is actually the sum of three independent Lévy processes, one of which is a Brownian motion with drift. When $\sigma > 0$ the resulting Lévy process will have a Brownian component and paths of infinite variation.

Cramér-Lundberg Process

An example of a finite activity, spectrally negative Lévy process is the Cramér-Lundberg process which satisfies

$$X_t = x + ct + \sum_{i=1}^{N_t}\xi_n, \qquad t \geq 0. \tag{2.8}$$

In (2.8) it is assumed that: $c > 0$; $N = \{N_t : t \geq 0\}$ is a Poisson process with parameter λ; and $\{\xi_n\}_{n\geq 1}$ is a sequence of negative, independent and identically distributed (i.i.d.) random variables with law μ. Also, it is assumed that ξ_n is independent of N for all n. In other words, X is a compound Poisson process with positive drift, started from x, that experiences decreases only in the form of jumps. If one thinks of c as a rate of incoming capital, and of the jumps as claims occurring at random times, one can appreciate why the Cramér-Lundberg process is often used to model the net wealth of an insurance company.

Using Definition 1 one can readily show that the Cramér-Lundberg process is in fact a Lévy process. As such, one can calculate the characteristic exponent of X as follows:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{izX_t}\right] &= \mathrm{e}^{izct}\mathbb{E}\left[\exp(iz\sum_{n=0}^{N_t}\xi_n)\right] \\
&= \mathrm{e}^{izct}\sum_{k=0}^{\infty}\mathbb{E}\left[\exp(iz\sum_{n=0}^{k}\xi_n)\,\middle|\, N_t = k\right]\mathbb{P}\left(N_t = k\right) \\
&= \mathrm{e}^{izct}\sum_{k=0}^{\infty}\left(\int_{(-\infty,0)} \mathrm{e}^{izx}\mu(dx)\right)^k \mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!} \\
&= \exp\left(izct + \lambda t\int_{(-\infty,0)} \mathrm{e}^{izx}\mu(dx) - \lambda t\right) \\
&= \exp\left(-t\left(-izc + \lambda\int_{(-\infty,0)} (1-\mathrm{e}^{izx})\mu(dx)\right)\right), \text{ thus} \\
\Psi(z) &= -izc + \lambda\int_{(-\infty,0)} \left(1-\mathrm{e}^{izx}\right)\mu(dx). \qquad (2.9)
\end{aligned}$$

From (2.9) the Lévy measure can easily be identified as $\lambda\mu(dx)$. Since μ is a probability measure and λ is assumed to be finite, the conclusion that the Cramér Lunberg process has finite activity follows naturally.

It is an established fact that any Lévy process that has paths of bounded variation must satisfy $\sigma = 0$ and have the property $\int_{(-1,1)} x\nu(dx) < \infty$. One can rearrange the Laplace exponent of such a process as follows

$$\begin{aligned}
\psi(z) &= \left(-a - \int_{\mathbb{R}\setminus\{0\}} x\mathbf{1}_{\{|x|<1\}}\nu(dx)\right)z + \int_{\mathbb{R}\setminus\{0\}}(\mathrm{e}^{zx}-1)\nu(dx) \\
&= \mathrm{d}z + \int_{\mathbb{R}\setminus\{0\}}(\mathrm{e}^{zx}-1)\nu(dx). \qquad (2.10)
\end{aligned}$$

One notices that the Laplace exponent for the Cramér-Lunberg process (if it were calculated from (2.9) for example) would have almost the same form, except for the support of the Lévy measure. In such a way, one can view any process with paths of finite variation as nothing other than a compound Poisson process with drift $\mathrm{d} = -a - \int_{\mathbb{R}\setminus\{0\}} x\mathbf{1}_{\{|x|<1\}}\nu(dx)$.

Hyper-exponential Lévy Processes

Hyper-exponential processes are a family of Lévy processes that have Lévy measures with densities of the form

$$\eta(x) = \mathbf{1}_{\{x>0\}} \sum_{i=1}^{N} a_i \rho_i \mathrm{e}^{-\rho_i x} + \mathbf{1}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i \mathrm{e}^{\hat{\rho}_i x}, \tag{2.11}$$

where $a_i, \rho_i, \hat{a}_i$, and $\hat{\rho}_i$ are positive numbers. It is not difficult to show that (2.11) satisfies integrability condition (2.1), so that one is assured of the existence of Hyper-exponential processes. In this document the convention $\sum_1^0 = 0$ is observed, which shows that one can use (2.11) to create a spectrally negative process simply by setting $N = 0$. Further, the form of the density η ensures that all Hyper-exponential processes have finite activity, and when $\sigma = 0$, also have finite variation. One can observe that the the Cramér-Lundberg process is a simple Hyper-exponential process when the jumps $\{\xi_n\}_{n\geq 1}$ are i.i.d. exponential random variables with densities $\mathbf{1}_{\{x<0\}}\gamma \mathrm{e}^{\gamma x}$. In that case, one obtains $\nu(dx) = \mathbf{1}_{\{x<0\}}\lambda\gamma\mathrm{e}^{\gamma x}dx$ which has the desired form.

2.3 The Scale Function $W^{(q)}$

The following will provide a brief introduction to scale functions and highlight some key results which will be used in later sections of this document. The results presented here and significantly more information on the theory of scale functions can be found in [16, 20, 22] and [8].

Before defining scale functions, it is useful to make a few comments about the properties of the Laplace exponents of spectrally negative Lévy processes. Conditions (2.1) and (2.4), the fact that $\nu(0,\infty) = 0$, and the the definition of the Laplace exponent given in (2.5) ensure that the Laplace exponent of a spectrally negative process exists for all $\mathrm{Re}(z) \geq 0$. In particular, it exists for $z \in \mathbb{R}$ such that $z \geq 0$. In this case, one can establish that ψ is strictly convex and tends to infinity as z tends to infinity, so that for $q \geq 0$ the quantity

$$\Phi(q) = \sup\{z \geq 0 : \psi(z) = q\}$$

is finite. With these concepts in place one can define the scale function $W^{(q)}(x)$.

Definition 3. *For a spectrally negative process X and any $q \geq 0$ the q-scale function is a function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for $x < 0$, and for $x \in [0, \infty)$, $W^{(q)}$ is the unique right continuous function whose Laplace transform is given by*

$$\int_0^\infty e^{-zx} W^{(q)}(x) dx = \frac{1}{\psi(z) - q}, \qquad z > \Phi(q). \tag{2.12}$$

Since q is usually specified, one can simply speak about the scale function rather than the q-scale function. One can observe that finding an explicit expression for $W^{(q)}$ may or may not be possible depending on the form of the Laplace exponent. However, for certain processes, the form of ψ leads to a tractable result for $W^{(q)}$.

The term "scale function" originated in describing the so-called one-sided and two-sided exit problems. These problems are concerned with describing how, when, and where a process X exits a strip or the half plane. The scale function appears in almost every identity that can be calculated for any of these problems. In particular, if

$$\tau_x^+ = \inf\{t > 0 : X_t > x\} \text{ and } \tau_x^- = \inf\{t > 0 : X_t < x\}$$

then a quantity of interest is $\mathbb{E}_x \left[\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_0^- > \tau_a^+\}} \right]$. It can be shown – see for example theorem 8.1 in [20] or Section 2.2 of [16] – that for $q \geq 0$ and $0 \leq x \leq a$

$$\mathbb{E}_x \left[\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_0^- > \tau_a^+\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)} . \tag{2.13}$$

The expression (2.13) is reminiscent of an identity for diffusions (see pg. 14 in [5]) in which the function playing the role of $W^{(q)}$ is termed the scale function.

In order to prove some of the forthcoming theorems it is necessary to describe the shape and smoothness properties of the scale function. In particular, [16] shows that $W^{(q)}$ has the following properties as x approaches 0 from above.

Lemma 2. *For all $q \geq 0$, $W^{(q)}(0) = 0$ if and only if X has unbounded variation. Otherwise, when*

X has bounded variation, $W^{(q)}(0) = 1/d$, where $d = -a - \int_{\mathbb{R}\setminus\{0\}} x\mathbf{1}_{\{|x|<1\}}\nu(dx)$. Further,

$$W^{(q)'}(0+) = \begin{cases} 2/\sigma^2 & \textit{when } \sigma \neq 0 \\ (\nu(-\infty,0)+q)/d^2 & \textit{when } \sigma = 0 \end{cases}$$

where the second case is understood to be $+\infty$ when $\nu(-\infty,0) = +\infty$.

Further properties of $W^{(q)}$ can be described using the following change of measure

$$\left.\frac{d\mathbb{P}_x^a}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathrm{e}^{a(X_t - x) - \psi(a)t}. \tag{2.14}$$

Under $\mathbb{P}^a$, X remains a spectrally negative Lévy process with Laplace exponent $\psi_a(z) = \psi(z + a) - \psi(a)$. Specifically, for $a = \Phi(q)$ the process $(X, \mathbb{P}_x^{\Phi(q)})$ has Laplace exponent $\psi_{\Phi(q)}(z) = \psi(z + \Phi(q)) - q$. Note that,

$$\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0 \ , \tag{2.15}$$

which follows from the fact that ψ is strictly convex, that $\psi(0) = 0$, that $q \geq 0$, and that ψ tends to infinity as $z \to +\infty$. Define the zero scale function

$$W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)} \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0) \ , \tag{2.16}$$

where $\underline{X} = \{\inf_{0 \leq s \leq t} X_s : t \geq 0\}$. Using the Wiener-Hopf factorization for spectrally negative Lévy processes one can show that the Laplace transform of $W_{\Phi(q)}(x)$ is equal to $1/\psi(\Phi(q))$ so that the definition is consistent with (2.12). Then, one can show that the following equality holds:

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x) \ . \tag{2.17}$$

From (2.15) it follows that $\left(X, \mathbb{P}_x^{\Phi(q)}\right)$ drifts to $+\infty$ so that $\lim_{x\to+\infty} \mathbb{P}_x^{\Phi(q)}(\underline{X}_\infty \geq 0) = 1$ and $\lim_{x\to+\infty} W_{\Phi(q)}(x) = 1/\psi'(\Phi(q))$. Since $W_{\Phi(q)}(x)$ is an increasing function this shows that $W_{\Phi(q)}(x) \leq 1/\psi'(\Phi(q))$ for all $x \geq 0$ which in turn suggests that $W^{(q)}(x)$ behaves like $\mathrm{e}^{\Phi(q)x}$ for large values of

x.

Before continuing, it is necessary to give the following definition.

Definition 4. *A function f is said to be completely monotone if*

(i) $f : [0,\infty) \to [0,\infty)$, f is continuous on $[0,\infty)$, and f is infinitely differentiable on $(0,\infty)$ so that

$$(-1)^n f^{(n)}(x) \geq 0 \text{ for } n = 0, 1, 2 \ldots, \text{ or}$$

(ii) $f(-x)$ satisfies (i).

Under the assumption that ν has no atoms (which will be the case henceforth) or that $\sigma > 0$, one can be certain that $W^{(q)}$ is continuously differentiable (see [8]). With this result and Definition 4 it is possible to give two important results by Loeffen [22, 23] which will be used several times in later sections.

Lemma 3. *If the Lévy measure ν of X has a completely monotone density, then $W^{(q)'}$ is strictly convex on $(0,\infty)$ for all $q > 0$.*

Lemma 4. *Under the same conditions as lemma 3 the scale function can be written as*

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} - f(x), \qquad x > 0$$

where f is a completely monotone function.

Now, define a^* as

$$a^* = \sup\left\{a \geq 0 : W^{(q)'}(a) \leq W^{(q)'}(x) \text{ for all } x \geq 0\right\},$$

and note that one can use (2.17) to show that $\lim_{x\to+\infty} W^{(q)'}(x) = +\infty$ so that $a^* < \infty$. From lemma 3 it follows that a^* is the unique point at which $W^{(q)'}$ attains its minimum when ν has a completely monotone density. That is, $W^{(q)'}$ is strictly decreasing on $(0, a^*)$ and strictly increasing on (a^*, ∞).

One final result concerning the scale function is needed to complete this introduction. The density of the potential measure $\hat{u}^{(q)}(x)dx$ of the dual process $\hat{X} = -X$ is defined as

$$\hat{u}^{(q)}(x)dx = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}(\hat{X}_t \in dx) \ .$$

The potential measure's relationship to the scale function is through its density and is given by the equality

$$\hat{u}^{(q)}(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} - W^{(q)}(x) \ . \tag{2.18}$$

Remembering that the scale function behaves like $\mathrm{e}^{\Phi(q)x}$ for large values of x it is not difficult to imagine that $\hat{u}^{(q)}(x)$ is bounded, a fact which can be established (see [16]). Note also that the results of lemma 4 show that $\hat{u}^{(q)}(x)$ is a completely monotone function when ν is completely monotone. Later sections dealing with numerical approximations of scale functions will make use of the form of the Laplace transform of $\hat{u}^{(q)}(x)$ which for $\Phi(q) > 0$ and $z \geq 0$ is given by

$$F^{(q)}(z) = \int_0^\infty \mathrm{e}^{-zx}\hat{u}^{(q)}(x)dx = \frac{1}{\psi'(\Phi(q))(z-\Phi(q))} - \frac{1}{\psi(z)-q} \ . \tag{2.19}$$

The closed form expression on the right hand side of (2.19) is obtained by applying the Laplace transform to the right hand side of (2.18) and completing the integration first for all $z > \Phi(q)$. Then, via an argument of analytic continuation, one can extend the result to the domain $z \geq 0$ (see [16]).

2.4 The Beta and Theta Families of Meromorphic Processes

A series of recent papers [14, 15, 17, 18] discusses a special class of Lévy processes called Meromorphic processes (M-processes). M-processes have the property that their Laplace exponents are meromorphic functions when extended to the complex plane (meaning the functions are analytic in the complex plane except for at most countably many poles). Of particular interest will be two families of M-processes called the beta and theta families. These families are each determined by sets of nine and ten parameters. By varying the parameters one can create processes with paths of infinite activity and infinite variation that have tractable Laplace exponents and explicit expressions for $W^{(q)}$.

To introduce the beta and theta families, first consider the series

$$\mathbf{1}_{\{x<0\}} \sum_{n=1}^{N} a_n \rho_n \mathrm{e}^{\rho_n x}, \tag{2.20}$$

where N can be infinite, a_n and ρ_n are strictly positive, and $\{\rho_n\}_{n=1}^{N}$ is a strictly increasing sequence such that $\rho_n \to +\infty$ as $n \to \infty$ when $N = \infty$. At the moment, consider only the case that $N = \infty$, and suppose that (2.20) converges uniformly on $(-\infty, -\varepsilon)$. A sufficient condition to ensure uniform convergence is given in the following proposition.

Proposition 5. *To achieve uniform convergence of the series in (2.20) it is sufficient to require that the series*

$$\sum_{n\geq 1} a_n \rho_n^{-2} \tag{2.21}$$

converges.

Proof. Indeed, one can verify that $\exp(-\rho_n \varepsilon) < \rho_n^{-3}$ for large enough N. To do so, note that repeated application of L'Hopital's rule gives $\lim_{x\to\infty} x^3 \exp(-x\varepsilon) = 0$. Thus, series (2.20) has larger summands than series (2.21) for only finitely many terms. The comparison test then gives the desired result. □

Proposition 5 helps to establish that one can use (2.20) to build a Lévy measure.

Proposition 6 (Proposition 1, [15])**.** *Assume that (2.21) is a convergent series. Then* $\nu(dx) = \mathbf{1}_{\{x<0\}} \sum_{n\geq 1} a_n \rho_n e^{\rho_n x} dx$ *must satisfy* $\int_{(-\infty,0)} (1 \wedge x^2) \nu(dx) < \infty$ *thereby demonstrating that* ν *is a Lévy measure.*

Proof.

$$
\begin{aligned}
\int_{-\infty}^{-\varepsilon} x^2 \nu(dx) &= \int_{-\infty}^{-\varepsilon} x^2 \sum_{n\geq 1} a_n \rho_n \mathrm{e}^{\rho_n x} dx \\
&= \sum_{n\geq 1} a_n \rho_n \int_{-\infty}^{-\varepsilon} x^2 \mathrm{e}^{\rho_n x} dx \\
&= \sum_{n\geq 1} a_n \rho_n^{-2} \int_{-\infty}^{-\rho_n \varepsilon} u^2 \mathrm{e}^{u} du \\
&< \sum_{n\geq 1} a_n \rho_n^{-2} \int_{-\infty}^{-\rho_1 \varepsilon} u^2 \mathrm{e}^{u} du \\
&< 2 \sum_{n\geq 1} a_n \rho_n^{-2}.
\end{aligned} \tag{2.22}
$$

The second equality results from interchanging integration and summation which is permissible since the series converges uniformly. The third equality follows from the change of variables $x \mapsto u = \rho_n x$. Together, the calculations in (2.22) show that $\int_{-\infty}^{-\varepsilon} x^2 \nu(dx)$ is increasing and bounded as $\varepsilon \to 0^-$ which demonstrates the convergence of the integral on $(-\infty, 0)$. As a result (2.1) is satisfied and (2.20) may be used as the density of a Lévy measure. Of course, the identical outcome can be achieved when $N < \infty$ – in which case the resulting Lévy measure is that of a hyper-exponential process – but with far less effort. $\square$

The form of the density of the measure ν has useful properties which are captured by the following two theorems. The proofs are reproduced here from [15].

Theorem 7. *Suppose a Lévy process X has Lévy measure $\nu(dx) = \mathbf{1}_{\{x<0\}} \sum_{n=1}^{N} a_n \rho_n e^{\rho_n x} dx$ where N can be infinite, a_n and ρ_n are strictly positive, and $\{\rho_n\}_{n=1}^{N}$ is a strictly increasing sequence such that $\rho_n \to +\infty$ as $n \to \infty$ when $N = \infty$. Also suppose that $\sum_{n\geq 1} a_n \rho_n^{-2}$ converges if $N = \infty$. Then X has a Laplace exponent ψ which is a real meromorphic function that has the following partial fraction decomposition*

$$
\psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n + z)}, \quad z \in \mathbb{C}. \tag{2.23}
$$

Proof. Using the form of the Laplace exponent derived following (2.6) one obtains

$$
\begin{aligned}
\psi(z) &= z\left(\int_{(-\infty,0)} x - x\mathbf{1}_{\{x>-1\}}\nu(dx) - a\right) + 1/2\sigma^2 z^2 + \int_{(-\infty,0)} (\mathrm{e}^{zx} - 1 - zx)\nu(dx) \\
&= z\left(\int_{-\infty}^{-1} x \sum_{n=1}^{N} a_n\rho_n \mathrm{e}^{\rho_n x} dx - a\right) + 1/2\sigma^2 z^2 + \int_{-\infty}^{0} (\mathrm{e}^{zx} - 1 - zx) \sum_{n=1}^{N} a_n\rho_n \mathrm{e}^{\rho_n x} dx \\
&= z\left(\sum_{n=1}^{N} a_n\rho_n \int_{-\infty}^{-1} x\mathrm{e}^{\rho_n x} dx - a\right) + 1/2\sigma^2 z^2 + \sum_{n=1}^{N} a_n\rho_n \int_{-\infty}^{0} (\mathrm{e}^{zx} - 1 - zx)\mathrm{e}^{\rho_n x} dx \\
&= \mu z + 1/2\sigma^2 z^2 + \sum_{n=1}^{N} a_n\rho_n \left(\frac{1}{z+\rho_n} - \frac{1}{\rho_n} + \frac{z}{\rho_n^2}\right) \\
&= \mu z + \frac{1}{2}\sigma^2 z^2 + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n + z)}
\end{aligned} \tag{2.24}
$$

When $N = \infty$ the interchange of summation and integration is again justified by uniform convergence. Additionally, it is straightforward to check that the term μ is finite given the assumptions. □

It is theorem 7 that gives M-processes their name. One notices that ψ has simple poles at the points $\{-\rho_n\}_{n=1}^{N}$. This observation and the result of the following theorem will be used repeatedly in later sections to obtain numerical solutions.

Theorem 8. *Assume that $q > 0$ and that the assumptions of theorem 7 hold. Then the equation $\psi(z) = q$ has negative solutions $\{-\zeta_n\}_{n=1}^{M}$ where $M = N$ when $\sigma^2 = 0$ and $N < \infty$, $M = N + 1$ when $\sigma^2 > 0$ and $N < \infty$, and $M = \infty$ when $N = \infty$. Further, $\{\zeta_n\}_{n=1}^{M}$ is a sequence of positive real numbers that satisfies the interlacing property*

$$
0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots . \tag{2.25}
$$

Proof. The proof requires repeated application of the intermediate value theorem. Let $f(z) = \psi(z) - q$. Now, $f(0) = -q < 0$ and $\lim_{z\to-\rho_1^+} f(z) = +\infty$. Since f is continuous there exists a point $c \in (-\rho_1, 0)$ such that $f(c) = 0$. A similar argument guarantees that roots exist between any two adjacent poles. When $N < \infty$ and $\sigma^2 > 0$ $(= 0)$ one can combine terms on the right hand side of (2.23) to obtain a polynomial of degree $N + 2$ $(N + 1)$ in the numerator, and a polynomial of degree N in the denominator. In short, when N is finite and $\sigma^2 > 0$ $(\sigma^2 = 0)$, f is a rational function with N simple poles, located at points $\{-\rho_n\}_{n=1}^N$, and $N+2$ $(N+1)$ real roots, located at points $\Phi(q)$, and $\{-\zeta_n\}_{n=1}^{N+1}$ $(\{-\zeta_n\}_{n=1}^{N})$. To verify that $-\zeta_{N+1} < -\rho_N$ when $\sigma^2 > 0$ simply notice that $\lim_{z\to-\infty} f(z) = +\infty$ and $\lim_{z\to-\rho_N^-} f(z) = -\infty$ and apply the intermediate value theorem as before. □

When N is finite and the underlying process is a hyper-exponential process, the previous two theorems identify the approximate location of all of the zeros of f and show that they are real. When $N = \infty$ the result is not as obvious. That is, it is not immediately clear that $\Phi(q)$, and $\{-\zeta_n\}_{n=1}^N$ comprise the entire set of zeros for f. For a proof that the result does in fact hold for $N = \infty$ consult [15]. Figure 1 can aid in understanding the previous discussion graphically. The figure shows an example of the graph of the Laplace exponent for a process in the theta family.

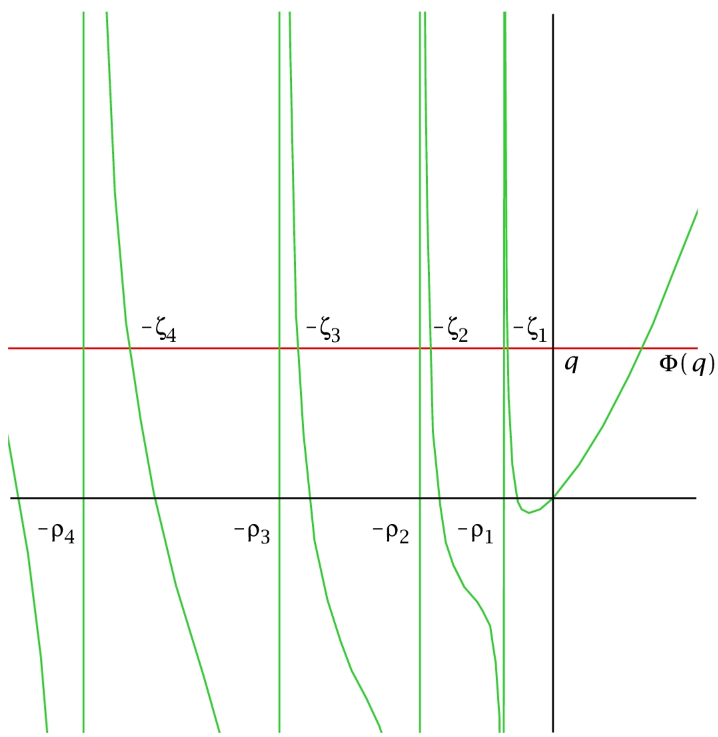


Figure 1: The Laplace exponent for a process in the theta family

With these general results established, one can now describe the theta and beta families in more detail. So far, the discussion has focused only on the case where $\nu(0,\infty) = 0$. It should be noted that both families can be extended to include Lévy measures supported on $\mathbb{R}\backslash\{0\}$. This is achieved by adding another series expression like (2.20), valid for $x > 0$, in the formula for the density of ν. In this document only the case where $\nu(0, \infty) = 0$ will be considered.

The family of theta processes derives its name from the Jacobi theta function θ_3 which has the form

$$\theta_3(z, q) = \sum_{-\infty}^{\infty} q^{n^2} \mathrm{e}^{2izn}. \tag{2.26}$$

Using (2.26) one can define $\Theta_0(x)$ as follows:

$$\Theta_0(x) = \theta_3(0, \mathrm{e}^{-x}) = \sum_{-\infty}^{\infty} \mathrm{e}^{-xn^2} = 1 + 2\sum_{n\geq 1} \mathrm{e}^{-xn^2}, \quad x > 0. \tag{2.27}$$

Inspired by (2.27), let

$$\Theta_k(x) = \delta_{k,0} + 2\sum_{n\geq 1} n^{2k} \mathrm{e}^{-xn^2}, \quad x > 0, \tag{2.28}$$

where $\delta_{k,0} = 1$ if $k = 0$ and $\delta_{k,0} = 0$ otherwise. With (2.28) one may then construct the density of the Lévy measure for the family of theta processes. This has the form

$$\eta_\lambda(x) = \mathbf{1}_{\{x<0\}} \frac{c}{\pi} \beta \mathrm{e}^{\alpha\beta x} \Theta_k(-x\beta), \tag{2.29}$$

where $c, \alpha, \beta > 0$, $0 < \lambda < 3$, and $k = \lambda - 1/2$. A quick test shows that one can write the density in the same form as (2.20) with

$$\begin{aligned} a_n &= (2\frac{c}{\pi}\beta n^{2\lambda-1})/(\beta(\alpha + n^2)) \text{ , and} \\ \rho_n &= \beta(\alpha + n^2), \end{aligned} \tag{2.30}$$

and that $\sum_{n\geq 1} a_n \rho_n^{-2}$ converges. Thus, (2.29) is a suitable density for defining a Lévy measure.

Examining the asymptotic behaviour of the density at 0 and $-\infty$ yields the following useful result.

Proposition 9 (Proposition 4, [15]). *The density η_λ satisfies*

$$\eta_\lambda(x) \sim \begin{cases} \Gamma(\lambda)\frac{c}{\pi}\beta^{1-\lambda}|x|^{-\lambda}, & \text{as } x \to 0^-, \text{ and} \\ 2\frac{c}{\pi}\beta e^{\beta(1+\alpha)x}, & \text{as } x \to -\infty. \end{cases} \tag{2.31}$$

Proof. The first asymptotic equivalence can be established by letting $h = \sqrt{-\beta x}$ and using the fact that a definite integral is the limit of a Riemann sum. That is, as $h \to 0^+$,

$$\begin{aligned} 2\frac{c}{\pi}\beta \mathrm{e}^{\alpha\beta x}\sum_{n\geq 1} n^{2k}\mathrm{e}^{n^2\beta x} &= 2\frac{c}{\pi}\beta \mathrm{e}^{\alpha\beta x} h^{-1-2k}\left(h\sum_{n\geq 1}(hn)^{2k}\mathrm{e}^{-(hn)^2}\right) \\ &= 2\frac{c}{\pi}\beta \mathrm{e}^{\alpha\beta x}(-\beta x)^{-\frac{1}{2}-k}\left(\int_0^\infty y^{2k}\mathrm{e}^{-y^2}dy + o(1)\right) \\ &= \frac{c}{\pi}\beta^{1-\lambda}\mathrm{e}^{\alpha\beta x}|x|^{-\lambda}\left(\int_0^\infty t^{k-\frac{1}{2}}\mathrm{e}^{-t}dt + o(1)\right) \\ &= \Gamma(\lambda)\frac{c}{\pi}\beta^{1-\lambda}|x|^{-\lambda}(1+o(1)). \end{aligned} \tag{2.32}$$

Additionally, one can check the behaviour of η_λ at negative infinity to see that

$$\begin{aligned} 2\frac{c}{\pi}\beta \mathrm{e}^{\alpha\beta x}\sum_{n\geq 1} n^{2k}\mathrm{e}^{n^2\beta x} &= 2\frac{c}{\pi}\beta \mathrm{e}^{\beta(1+\alpha)x}\sum_{n\geq 1} n^{2k}\mathrm{e}^{(n^2-1)\beta x} \\ &= 2\frac{c}{\pi}\beta \mathrm{e}^{\beta(1+\alpha)x}\left(1+\sum_{n\geq 2} n^{2k}\mathrm{e}^{(n^2-1)\beta x}\right) \\ &= 2\frac{c}{\pi}\beta \mathrm{e}^{\beta(1+\alpha)x}(1+o(1)). \end{aligned} \tag{2.33}$$

□

Now, notice that when $\lambda \geq 1$ the integral $\int_{-\varepsilon}^0 |x|^{-\lambda}dx = \infty$ for every $\varepsilon > 0$. Thus, by modifying values of λ one is able to change the Lévy measure to produce processes of infinite activity and infinite variation. That is, provided $\sigma = 0$, if $\lambda \in (0,1)$ one defines a family of process of finite activity and finite variation. For $\lambda \in [1,2)$ the resulting processes will have infinite activity and finite variation. Finally, for $\lambda \in [2,3)$ the processes will have both infinite activity and infinite variation.

For several values of λ one can obtain manageable expressions for the Laplace exponent ψ from the series expression given in (2.23). For details of this procedure see [15]. This thesis will focus on two infinite activity families, one with $\lambda = 3/2$ that has paths of bounded variation when $\sigma = 0$, and one with $\lambda = 5/2$ that has paths of unbounded variation. Their respective Laplace exponents are given by

- θ-process with parameter $\lambda = 3/2$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z - c\sqrt{\alpha + z/\beta}\coth\left(\pi\sqrt{\alpha + z/\beta}\right) + c\sqrt{\alpha}\coth\left(\pi\sqrt{\alpha}\right), \text{ and} \tag{2.34}$$

- θ-process with parameter $\lambda = 5/2$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + c\left(\alpha + z/\beta\right)^{\frac{3}{2}} \coth\left(\pi\sqrt{\alpha + z/\beta}\right) - c\alpha^{\frac{3}{2}}\coth\left(\pi\sqrt{\alpha}\right). \tag{2.35}$$

Formulas (2.34), (2.35), and (2.30), will play a significant part in the computations performed in later sections.

The formulas of the Laplace exponents for members of the beta family have similar derivations as those of their theta family counterparts. The beta family Lévy measure has density

$$\eta_\lambda(x) = c\beta\frac{\mathrm{e}^{(1+\alpha)\beta x}}{(1-\mathrm{e}^{\beta x})^\lambda}, \quad x < 0. \tag{2.36}$$

Using the binomial series one can expand the denominator of (2.36) to give a series expression like (2.20) that satisfies the conditions of theorem 7. One finds that the resulting Laplace exponent will have poles

$$\rho_n = \beta(\alpha + n). \tag{2.37}$$

Further, much like the theta family η_λ satisfies

$$\eta_\lambda(x) \sim \begin{cases} |x|^{-\lambda}, & \text{as } x \to 0^-, \text{ and} \\ \mathrm{e}^{\beta(1+\alpha)x}, & \text{as } x \to -\infty, \end{cases} \tag{2.38}$$

up to some multiplicative constants. Therefore, $0 < \lambda < 3$ as before, and λ controls the activity and variation of the process in the same fashion as it does for the theta family. For the beta family, however, it is possible to find suitable Laplace exponents for a range of λ values (see [14] for details). In particular, this thesis will use the following expression, primarily for cases where $\lambda \in (1,2)$ or $\lambda \in (2,3)$:

- β-process with parameter $\lambda \in (0,3)\backslash\{1,2\}$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + c\mathrm{B}\left(1+\alpha+z/\beta, 1-\lambda\right) - c\mathrm{B}\left(1+\alpha, 1-\lambda\right). \tag{2.39}$$

From (2.39) one can appreciate that the family is aptly named since $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function.

The nature of the Laplace exponents derived above allows one to calculate an explicit expression for the scale function $W^{(q)}$. The full proof of this result, and the form of the scale function can be found in [18].

Proposition 10. *For $q \geq 0$ hyper-exponential processes and processes with Laplace exponents given by (2.34), (2.35), and (2.39) have scale functions of the form*

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \sum_{n=1}^{N} \frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)}, \tag{2.40}$$

where $N = \infty$ for (2.34), (2.35), and (2.39). In these cases the series is uniformly convergent on sets of the form (ε, ∞), where $\varepsilon > 0$.

Proof (sketch). To give an idea of the proof consider a general M-process with Laplace exponent given by (2.23). Assume that N is finite and that $\sigma = 0$ so that the function $f(z) = 1/(\psi(z) - q)$ is rational with N simple roots, $\{-\rho_n\}_{n=1}^N$, and $N+1$ simple poles, $\Phi(q)$ and $\{-\zeta_n\}_{n=1}^N$. From complex analysis it is known that f has residues given by $1/\psi'(r)$, where r is a pole of f, and a partial fraction decomposition of the form

$$f(z) = \frac{1}{\psi'(\Phi(q))(z - \Phi(q))} + \sum_{n=1}^{N} \frac{1}{\psi'(-\zeta_n)(z + \zeta_n)}. \tag{2.41}$$

To recover $W^{(q)}(x)$ one can apply the inverse Laplace transform to (2.41) to obtain (2.40). This proves the proposition for hyper-exponential processes. Further analysis in [18] shows that (2.40) is valid for the remaining three cases as well. See section 4.2 for the discussion on uniform convergence.

□

3 The Optimal Dividend Problem

3.1 Overview

The following problem was proposed by De Finetti [9] in a 1957 paper: Is there an optimal strategy π that maximizes the expected value of the discounted dividends paid to an insurance company's shareholders? The solution to this problem depends on how one models the company's gross wealth process $X = \{X_t : t \geq 0\}$ and what restrictions one imposes on π. The problem's simplest form assumes X is a discrete time random walk on the integers and restricts dividend policies to be integer valued as well. This was the problem first considered by Shubik and Thompson [26] and Miyasawa [24]. More recent papers have framed the problem in a continuous time setting by modelling X as a Brownian motion with positive drift [2], or as a Cramér-Lundberg process [13]. This has provided the impetus to study the problem in the more general spectrally negative setting. That is, recent papers have derived results for the general case, as opposed to approaching the problem from the perspective that X is a particular process.

In either the specific or general scenario, demonstrating optimality of a particular strategy requires roughly the same method of proof. Initially, one defines a set of admissible dividend strategies Π. The term admissible refers to certain technical conditions that narrow the problem; details will be supplied below as needed. Generally, members of Π will have the form $\pi = \{L_t^\pi : t \geq 0\}$ which denotes a non-negative, non-decreasing, left-continuous process that is adapted to the filtration $\mathbb{F}$ of X. Here, L_t^π represents the cumulative dividends paid up to time t. Then, if $U_t^\pi = X_t - L_t^\pi$ is the net wealth of the firm at time t, $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ is the time until ruin under strategy π, $x \geq 0$ is capital of the firm when $t = 0$, and $q > 0$ is a discount rate, one can define the value function

$$v_\pi(x) = \mathbb{E}_x \left[\int_{[0,\sigma^\pi]} \mathrm{e}^{-qt} dL_t^\pi \right]. \tag{3.1}$$

The aim in all cases is first to demonstrate that there exists a strategy π^* such that $v_{\pi^*}(x) = v_*(x) = \sup_{\pi\in\Pi} v_\pi(x)$ for all $x > 0$, second to find the form of π^*, and finally to find a computable formula for v_{π^*}.

The literature [2,3,9,12,13,19,22–24,26] shows that optimal strategies are typically of the *barrier* or *band* type. Band strategies define a partition of the state space of the π-controlled process U^π and vary the rate of dividend payout on intervals formed by the points in the partition. For example, consider partitioning the state space of U^π with points $a < b < c$. A strategy that pays no dividends when U_t^π is below point a, pays dividends at rate $r(U_t^\pi)$ when $U_t^\pi \in (a, b)$, and pays no dividends when $U_t^\pi > c$ is an example of a band strategy. A barrier strategy is a simple band strategy that has only one point, appropriately named the barrier, in the partition of the state space of U^π. One of the most common optimal strategies is a reflection strategy.

Definition 5. *A reflection strategy is a barrier strategy with barrier b that pays no dividends when $U_t^\pi < b$ and pays $U_t^\pi - b$ when $U_t^\pi \geq b$. Under a reflection strategy $L_t^\pi = \max\left[\left(\sup_{0\leq s\leq t} X_s - b\right), 0\right]$.*

One can argue that a reflection strategy is the most intuitively obvious optimal strategy. That is, without any mathematical analysis management is likely to use a similar strategy in order to optimize dividends. When the wealth of the firm is low, management will likely choose to cease paying dividends in order to avoid bankruptcy. At times when the level of wealth is high and exceeds suitable investment options, management will choose to return all excess capital to the shareholders.

These ideas are illustrated with an example for the case of the random walk. The example is adapted from [26]. Unless otherwise stated, the reader may assume all theorems, lemmas, facts and associated proofs in the next section are taken from [26].

3.2 Example: Optimal Policy for the Random Walk

The random walk is a stochastic process in discrete time that takes values on the integers. For each $n \in \mathbb{N}$, $X_{n+1} = X_n - 1$ or $X_{n+1} = X_n + 1$ with probability $1 - p$ and p respectively. The set of admissible strategies Π, will include any strategy paying dividends in integer valued amounts. Strategies are permitted to prescribe the withdrawal of all remaining capital, but when the net wealth process reaches 0, either as the result of a dividend payment, or as a result of an evolution of the gross wealth process, the company is presumed to be bankrupt and no further dividends are paid. A dividend strategy π will be defined by a function ℓ^π from the positive to the non-negative integers that depends only on U_n^π. (Therefore only so-called simple strategies will be considered.) That is, $\ell^\pi(U_{n+1}^\pi) = L_{n+1}^\pi - L_n^\pi = \Delta L_n^\pi$, where $U_0^\pi = x$ and $\ell^\pi(x) = L_0^\pi - 0 = \Delta L_0^\pi$. Then

$U^\pi_{n+1} = U^\pi_n \pm 1 - \ell^\pi(U^\pi_n)$ and the value function becomes

$$v_\pi(x) = \mathbb{E}_x\left[\sum_{n=0}^{\sigma^\pi - 1} \Delta L^\pi_n q^n\right] = \mathbb{E}_x\left[\sum_{n=0}^{\sigma^\pi - 1} \ell^\pi(U^\pi_n) q^n\right] . \tag{3.2}$$

Note that in this discrete case it is no longer assumed that π is left-continuous. Also, the definition of σ^π changes slightly to $\sigma^\pi = \inf\{n > 0 : U^\pi_n \leq 0\}$. Finally, a technical condition is imposed so that the state space of U^π remains finite for all $\pi \in \Pi$. For some finite, suitably large $L \in \mathbb{N}$, $\ell^\pi(L) \geq 1$ for all $\pi \in \Pi$. Thus, starting capital x must also be less than or equal to L. There is some financial justification for this technical condition since a company will not accumulate wealth indefinitely without returning capital to its investors.

To see that (3.2) exists, consider the case in which $p = 1$. Then, π^* is the strategy that withdraws all but one unit of the starting capital in the first period, and immediately pays out every gain thereafter. That is,

$$v_{\pi^*}(x) = (x-1) + \sum_{n=0}^{\infty} q^n = (x-1) + \frac{q}{1-q} . \tag{3.3}$$

It is assumed that $q > 1/2$ so that (3.3) is strictly greater than x (see Lemma 11 for discussion). For any other strategy π or values of p, the series inside the expectation on the right hand side of (3.2) is bounded by (3.3). Since the series consists only of non-negative terms, its sequence of partial sums forms a monotonic sequence which must converge. Thus, the expectation in (3.2) is finite, which shows that $v_\pi(x)$ is finite as well.

It is reasonable to expect that investors would decline to invest under the condition that either p or q is too small. Therefore, all further analysis will proceed under the assumption that $pq > 1/2$. The following lemma provides justification for this assumption.

Lemma 11 (Stated but not proven in [26])**.** *If $pq \leq 1/2$ then $x \geq v_\pi(x)$ for all $x \in [0, L]$ and $\pi \in \Pi$. That is, if $pq \leq 1/2$ then immediate withdrawal of the starting capital is the best course of action.*

Proof. Consider the finite horizon problem with expiry time $\tau < \infty$. That is, for all $n \geq \tau$ the company pays no dividends. Then, one can define the value function as

$$v_\pi(\tau, x) = \mathbb{E}_x \left[\sum_{n=0}^{(\sigma^\pi - 1)\wedge\tau} \ell^\pi(U_n^\pi) q^n \right].$$

Clearly, when $\tau = 0$ the only choice is to withdraw all capital. If $\tau = 1$ one can pay dividends of size u where $0 \leq u \leq x$. Once the dividend is paid, the net wealth process will be incremented, and the remaining capital will be paid since the expiry time has been reached. Thus, there are x possible strategies, one for each possible dividend payment at time 0. For strategies with payments $0 \leq u < x$ the value function will be

$$\begin{aligned} v_\pi(\tau, x) &= u + q(p(x-u+1) + (1-p)(x-u-1)) \\ &= u + q(p - (1-p) + (x-u) \\ &= u + q(x-u) + q(2p-1), \end{aligned} \tag{3.4}$$

while for the strategy that pays all capital immediately the value function will be

$$v_\pi(\tau, x) = x.$$

From the assumptions one can calculate

$$\begin{aligned} u + q(x-u) + q(2p-1) &\leq u + q(x-u) + (1-q) \\ &= (1-q)u + qx + (1-q) \\ &\leq (1-q)(x-1) + qx + (1-q) \\ &= (1-q)x + qx = x. \end{aligned}$$

Thus the optimal course of action in the one period problem is to immediately pay the entire starting capital as a dividend. By induction one can extend this result to all $\tau \in \mathbb{N}$. Thus, one obtains

$$x \geq \mathbb{E}_x \left[\sum_{n=0}^{(\sigma^\pi - 1)\wedge\tau} \ell^\pi(U_n^\pi) q^n \right] \text{ for all } x \geq 0 \text{ and } \pi \in \Pi. \tag{3.5}$$

Taking the limit of both sides as $\tau \to \infty$ and applying the dominated convergence theorem on the right hand side then gives the desired result.

□

From the previous lemma and the discussion so far, it is clear that the aim is to find an optimal strategy. Formally, an *optimal* strategy π is one that satisfies $v_\pi(x) \geq v_{\pi'}(x)$ for all $x \geq 0$ and $\pi' \in \Pi$. The existence of an optimal strategy is not obvious. However, under the condition that $\sup_{\pi \in \Pi} v_\pi(x) < \infty$ for all $x \in [0, L]$, and that the state space of the controlled process U^π is finite for all $\pi \in \Pi$, [10], pq. 138 shows that a simple optimal strategy exists. [1]

A *special optimal strategy* is an optimal strategy that satisfies the following property: For $y < L$, $\ell^\pi(y) = 0$ if, and only if,

$$q(1-p)v_\pi(y-1) + qpv_\pi(y+1) > \max_{1 \leq a \leq y} [a + v_\pi(y-a)]. \tag{3.6}$$

Let S denote the set of special optimal strategies. The significance of condition (3.6) will be explained following the next lemma.

Lemma 12. *For $x < L$ the value function of an optimal strategy π must satisfy the following functional equation, which one recognizes as a variant of the Bellman equation.*

$$v_\pi(x) = \max \left[q(1-p)v_\pi(x-1) + qpv_\pi(x+1), \max_{1 \leq a \leq x} [a + v_\pi(x-a)] \right] . \tag{3.7}$$

For $x = L$ this simplifies to

$$v_\pi(x) = \max_{1 \leq a \leq x} [a + v_\pi(x-a)] . \tag{3.8}$$

Proof. The statement can be verified by arguing that a firm can either pay a dividend a out of its starting capital $x < L$ or proceed without an initial payment. In order for v_π to satisfy (3.7) the

[1] The existence of an optimal strategy is necessary to complete the remainder of this example. The authors of [26] do not limit the state space of U^π, but provide an incomplete proof of existence. The author of this thesis has not been able to obtain or formulate another proof that satisfactorily resolves the issue of existence for the unlimited case. Typically, in demonstrating optimality, one demonstrates that a strategy is optimal if and only if it satisfies a Hamilton-Jacobi-Bellman condition. Thus one finds a candidate strategy, often with little prior justification, and shows that the strategy satisfies the Hamilton-Jacobi-Bellman condition and must therefore be optimal. This example is included because it shows another method of demonstrating optimality. The advantage of this approach is that one develops a clear understanding as to why the strategy is chosen.

strategy π must make an initial choice that maximizes $v_\pi(x)$. Thus if v_π does not satisfy (3.7), a strategy that pays the maximizing initial dividend (this could mean paying no initial dividend at all) and then adopts the same strategy as π will have a greater value. The same argument holds for $x = L$, except in this case all strategies must make an initial payment. □

Now one can see that the definition of the special optimal strategy removes ambiguity when searching for a strategy whose value function satisfies (3.7). That is, suppose that for some optimal strategy π there exists one or more $1 \leq a \leq x$ such that the equality $q(1-p)v_\pi(x-1)+qpv_\pi(x+1) = a+v_\pi(x-a)$ holds. In this case v_π will satisfy (3.7) whether a dividend of amount a is paid or not. To determine a unique choice of optimal strategy, condition (3.6) is imposed.

The following facts are direct consequences of lemma (12), or of the properties of special optimal strategies. For $\pi \in S$:

1. v_π is a monotone strictly increasing function of x; and
2. if y is such that $\ell^\pi(y) = 0$, and $\ell^\pi(y+1) > 0$, then $\ell^\pi(y+1) = 1$.

Proof: Fact 1. From (3.7) it follows that $v_\pi(x) \geq 1 + v_\pi(x-1)$, so that $v_\pi(x) > v_\pi(x-1)$. □

Proof: Fact 2. Consider a dividend strategy that pays two units of wealth when $U_n^\pi = y + 1$. Suppose that the dividend payment is made in one unit instalments, prior to the next step in the wealth process. After the first payment the net wealth of the company is y so that under the given assumptions, no further dividend payments should be made. However, in order to pay a dividend of two units an additional payment is necessary. This intuitive argument indicates that a strategy that pays a dividend in excess of one unit of wealth when $U_n^\pi = y + 1$ is somehow inconsistent. For $y < L$, let $G(y) = q(1-p)v_\pi(y-1) + qpv_\pi(y+1)$, and note that one can re-write (3.7) as $v_\pi(y) = \max_{0\leq a\leq y}[a + G(y-a)]$. Thus if $\ell^\pi(y) = 0$, (3.6) guarantees that

$$\begin{aligned} v_\pi(y) &= \max_{0\leq a\leq y}[a + G(y-a)] \\ &= G(y)\ . \end{aligned} \tag{3.9}$$

Additionally, suppose $y+1 < L$, then

$$\begin{aligned} v_\pi(y+1) &= \max_{0\le a\le y+1}[a + G(y+1-a)] \\ &= \max\left[G(y+1), \max_{0\le a\le y}[1+a+G(y-a)]\right] \ . \end{aligned} \tag{3.10}$$

From (3.9), and (3.10) one obtains $v_\pi(y+1) = \max[G(y+1), 1+G(y)]$. Since it is assumed that $\ell^\pi(y+1) > 0$, one must conclude that $v_\pi(y+1) = 1+G(y)$ from which it follows that $\ell^\pi(y+1) = 1$.

If $y+1 = L$ one can obtain the same result by rewriting (3.10) as follows:

$$\begin{aligned} v_\pi(y+1) &= \max_{1\le a\le y+1}[a + G(y+1-a)] \\ &= \max\left[1+G(y), \max_{1\le a\le y}[1+a+G(y-a)]\right] && (3.11) \\ &= 1+G(y) \ . && (3.12) \end{aligned}$$

The last equality follows from (3.9) since, $a + G(y-a) < G(y)$ for all $1 \le a \le y$. □

Fact 2 seems to suggest that optimal strategies are reflection strategies. The following lemma establishes this fact.

Lemma 13 (Proof modified from [26])**.** *Let $\pi \in S$ and let b be the smallest positive integer such that $\ell^\pi(b+1) > 0$, then for all $y \ge b$, $\ell^\pi(y) = y - b$.*

Proof. First suppose that for some $y > b$, $\ell^\pi(y) = a > y - b$. Then, $b > y - a$ and

$$v_\pi(y-a) + a > v_\pi(b) + (y-b)$$

$$\Leftrightarrow \quad v_\pi(y-a) + (b-(y-a)) = v_\pi(b-(b-(y-a))) + (b-(y-a)) > v_\pi(b) \ .$$

This contradicts the fact that π is a special optimal strategy. An intuitive argument similar to the one made in the proof of Fact 2 can also be made here. If a dividend greater than $y-b$ is paid when the net wealth of the company is y, then it stands to reason that this can occur in instalments prior to the next step in the wealth process. Once the instalments total $y-b$ units of wealth, the strategy dictates that no further instalments should be paid since the net wealth of the company is now exactly b. Thus, the proposed strategy is inconsistent as before.

Next, suppose that for some $y > b$, $\ell^\pi(y) = a < y - b$. Then, there exists at least one $y' > b$ for which $\ell^\pi(y') = 0$. By intuition, one can recognize that $y - a$ should have the property $\ell^\pi(y - a) = 0$. Otherwise, if the strategy calls for the payment of a dividend when the net wealth of the company is $y - a$, then a dividend payment of a out of net wealth y would immediately be followed by another dividend payment prior to the next step in the wealth process. This implies that the dividend strategy actually calls for a dividend payment greater than a when the net wealth of the company is y. To make this precise, assume $\ell^\pi(y - a) = k > 0$. Then,

$$\begin{aligned} v_\pi(y) &= a + v_\pi(y - a) \\ &= a + \max_{0 \le j \le y-a} [j + G(y - a - j)] \\ &= a + k + G(y - a - k) \end{aligned}$$

which implies that $\ell^\pi(y) > a$ contrary to the assumption. Thus, $\ell^\pi(y - a)$ must be zero, which establishes the existence of y'.

Let M be the smallest integer greater than b such that $\ell^\pi(M) = 0$. Employing the same argument used to establish fact 2, and the fact that $\ell^\pi(L) \ge 1$, one can show that there exists a $y \ge M$ such that $\ell^\pi(y) = 0$ and $\ell^\pi(y + 1) = 1$. If N is the smallest such y, then ℓ^π has the following form:

$$\ell^\pi(y) = \begin{cases} 0 & \text{for } y \le b, \\ y - b & \text{for } b < y < M, \\ 0 & \text{for } M \le y \le N, \\ 1 & \text{for } y = N + 1. \end{cases}$$

With this one can show that π is not optimal which provides a contradiction. To do so consider the strategy π' which is given by

$$\ell^{\pi'}(y) = \begin{cases} 0 & \text{for } y \leq b, \\ y-b & \text{for } b < y < M-1, \\ 0 & \text{for } M-1 \leq y \leq N-1, \\ 1 & \text{for } y = N. \end{cases}$$

Note that b and $M-1$ may be adjacent integers. This special case will be treated separately after the following comments. For now, suppose $M-1 > b+1$ and notice that when $x = N$ the strategy π' immediately pays a dividend of one unit, and π pays nothing. Thus, $L_0^{\pi'} = 1$ whereas $L_0^{\pi} = 0$. At this point, the net wealth of the firm under strategy π' is $N-1$ while under π it remains N. Then, $L_n^{\pi'} - L_n^{\pi} = 1$ and $U_n^{\pi} - U_n^{\pi'} = 1$ until such time m when $U_m^{\pi} = M-1$, or equivalently, $U_m^{\pi'} = M-2$. If $m < \infty$, strategy π pays a dividend of $(M-1)-b$ while π' will pay $(M-2)-b$. Following this payment, $U_n^{\pi} = U_n^{\pi'}$ and $L_n^{\pi} = L_n^{\pi'}$ for all $n > m$ so that the strategies are equal beyond time m. That is, $L_n^{\pi'} - L_n^{\pi} = 1$ for $0 \leq n < m$, and $L_n^{\pi'} = L_n^{\pi}$ for $n \geq m$. If $m = \infty$ then $L_n^{\pi'} - L_n^{\pi} = 1$ for all $n \geq 0$. Thus dividend payments are equal under both strategies, except for the first payment, and possibly one additional payment at time m. More precisely, $\ell^{\pi'}(x) - \ell^{\pi}(x) = 1$ and if $m < \infty$ then $\ell^{\pi}(U_m^{\pi}) - \ell^{\pi'}(U_m^{\pi'}) = ((M-1)-b) - ((M-2)-b) = 1$. From this, and the fact that $q < 1$, one can conclude that when $x = N$

$$\sum_{n=0}^{\sigma^{\pi'}-1} \ell^{\pi'}(U_n^{\pi'})q^n - \sum_{n=0}^{\sigma^{\pi}-1} \ell^{\pi}(U_n^{\pi})q^n \geq 1 - q^m > 0. \tag{3.13}$$

The properties of integration and (3.6) lead to the conclusion that $v_\pi(N) < v_{\pi'}(N)$ which shows that π is not optimal. This is a contradiction, which proves that y' does not exist and that $\ell^{\pi}(y) = y - b$ for all $y > b$. For a graphical interpretation of these arguments consult Figure 2. The left column of the figure shows a scenario where m is beyond the time-line depicted by the graph, while the right column shows m within the time-line. Processes U^{π}, L^{π}, $U^{\pi'}$, and $L^{\pi'}$ are labelled U,L,U', and L' respectively.

For the special case where $b+1 = M-1$, notice that one cannot make the same arguments, because in this case, π' reduces to a strategy that pays a dividend only at N. The arguments hold with a slightly altered strategy π'' that behaves according to π' until the process $U^{\pi''}$ takes the value

$M - 2 = b$. From this point, π'' behaves according to π. Although the strategy π'' is no longer simple, the existence of a simple optimal strategy shows that $v_\pi(N) < v_{\pi''}(N)$ is still a contradiction.

□

Lemma 13 shows that all special optimal strategies are of the reflection type. If only one such special optimal strategy exists, then it will be the optimal strategy for the random walk problem. Theorem 14 follows from the facts already established.

Theorem 14 (Theorem 1 in [26])**.** *There is a unique special optimal strategy π^* defined as follows: there exists $b^* > 0$ such that $\ell^{\pi^*}(y) = 0$ if $y < b^*$ and $\ell^{\pi^*}(y) = y - b^*$ if $y \geq b^*$.*

Now it is possible to find an explicit formula for v_{π^*} by solving

$$v_{\pi^*}(x) = qpv_{\pi^*}(x+1) + q(1-p)v_{\pi^*}(x-1) \quad \text{for } x \leq b^* \tag{3.14}$$

subject to the conditions:

$$v_{\pi^*}(0) = 0 \ , \tag{3.15}$$

$$v_{\pi^*}(b^*+1) = v_{\pi^*}(b^*) + 1 \ . \tag{3.16}$$

Let $v_{\pi^*} = \alpha^x$, and note that this function satisfies (3.14) when

$$\alpha = \frac{1 \pm \sqrt{1 - 4q^2p(1-p)}}{2qp} \ .$$

If one denotes the roots by α_1 and α_2 one obtains the general solution to (3.14):

$$v_{\pi^*}(x) = C_1\alpha_1^x + C_2\alpha_2^x \ .$$

Using (3.15) and (3.16) yields:

$$C_1 = -C_2$$

and,

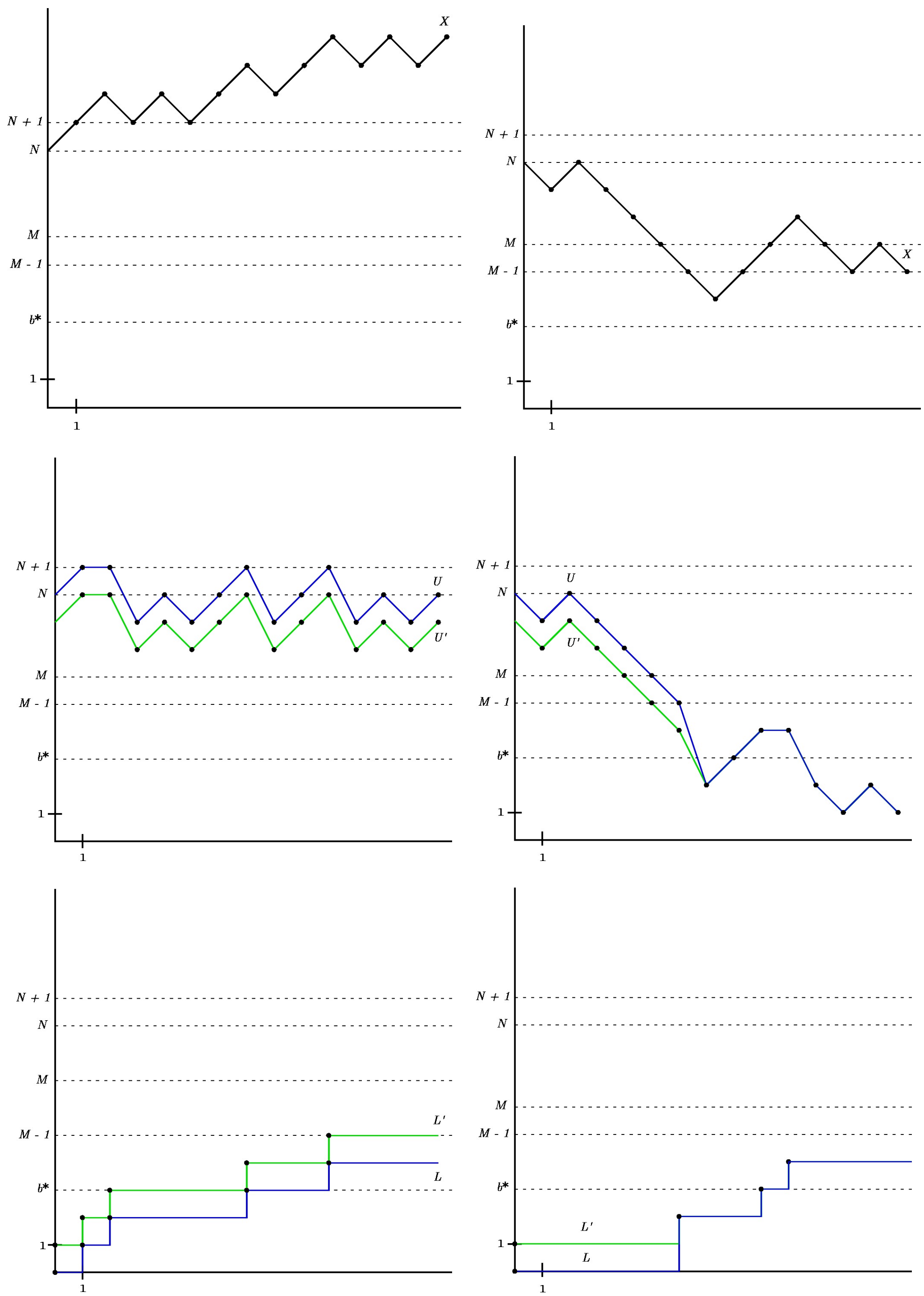


Figure 2: Graphical demonstration that π is not optimal when starting capital is N. The left-hand side depicts the case where m is beyond the given time-line, while the right-hand side shows m within the graph's time-line. Processes U^{π}, L^{π}, $U^{\pi'}$, and $L^{\pi'}$ are labelled U,L,U', and L' respectively.

$$C_1 = \frac{1}{\left(\alpha_1^{b^*+1} - \alpha_2^{b^*+1}\right) - \left(\alpha_1^{b^*} - \alpha_2^{b^*}\right)} \ .$$

Thus, one arrives at the following formula for v_{π^*}:

$$v_{\pi^*}(x) = \begin{cases} C_1\left(\alpha_1^x - \alpha_2^x\right) & x \leq b^* \\ (x - b^*) + v_{\pi^*}(b^*) & x > b^* \end{cases}$$

Remark Note that in order for b^* to be the optimal barrier, it must maximize C_1. For the purposes of this discussion, let C_1 be a function of the variable $b \in [0, L]$ and let b^* be the value that maximizes C_1. Observe that,

$$\frac{dC_1}{db} = \frac{\alpha_2^b\left(\alpha_2 - 1\right)\ln(\alpha_2) - \alpha_1^b\left(\alpha_1 - 1\right)\ln(\alpha_1)}{\left(\left(\alpha_1^{b+1} - \alpha_1^b\right) - \left(\alpha_2^{b+1} - \alpha_2^b\right)\right)^2},$$

and that,

$$\min_{(p,q)} \alpha_1 = 1, \text{ while } \max_{(p,q)} \alpha_2 = 1, \text{ for } (p,q) \in \left[{}^1/_2, 1\right] \times \left[{}^1/_2, 1\right],$$

where the exrema occur only at the boundaries of $\left[{}^1/_2, 1\right] \times \left[{}^1/_2, 1\right]$. Specifically, the maxima and minima occur only when $p = 1/2$ or $p = 1$. In these cases it has been shown that the barrier b^* is either 0 or 1 for all $L \in \mathbb{N}$. Since extrema occur only at the boundaries, $\alpha_1 > 1$ on $A = \left({}^1/_2, 1\right) \times \left[{}^1/_2, 1\right]$ so that

$$f(b) = \alpha_1^b\left(\alpha_1 - 1\right)\ln(\alpha_1)$$

is an increasing and strictly positive function. Similarly, for $(p,q) \in A$, $0 < \alpha_2 < 1$, so that

$$g(b) = \alpha_2^b\left(\alpha_2 - 1\right)\ln(\alpha_2)$$

is decreasing and strictly positive. Now consider two cases:

1. $f(0) \geq g(0)$, and

2. $f(0) < g(0)$.

In the first case, $C_1'(b) < 0$ for all $b > 0$. This shows that C_1 is a decreasing function on $[0, L]$ which is maximized when $b = 0$. In other words, the optimal barrier b^* is equal to 0 which means that optimal strategy immediately withdraws the entire starting capital. Notice that neither $f(0)$ nor $g(0)$ depend on L so that this conclusion holds for all $L \in \mathbb{N}$.

In the second case $C_1'(b) > 0$ until $f(b) = g(b)$. Since $\lim_{b\to\infty} f(b) = \infty$ and $\lim_{b\to\infty} g(b) = 0$ there exists a unique finite c such that $f(c) = g(c)$. Note that C_1 is increasing for all $b < c$ and decreasing for all $b > c$. Thus if $L \leq c$, the optimal barrier b^* must equal L. However, for all $L > c$, the optimal barrier b^* must be equal to an integer j which is either adjacent or equal to c. This shows that the value of b^* only depends on L when $L \leq c$, which implies that the barrier strategy with $b^* = j$ is optimal for all but finitely many values of $L \in \mathbb{N}$.

This discussion provides strong evidence, although not a rigorous proof, that the result also holds when the state space of U^π is unbounded.

3.3 Evolution of the Problem

There are a number of natural extensions to the case where X is modelled by a random walk. The first is to generalize the random walk by allowing X to be a discrete time, discrete space Markov process. In this model the process is assumed to have stationary transition probabilities with state space equal to the integers. This case was considered by Miyasawa [24], who found that under the condition that the state space of the process is bounded below, the optimal strategy always exists and is given by a band strategy. The exact nature of this strategy is specified in the following theorem.

Theorem 15. *Under the conditions established in the previous paragraph there exists a unique finite sequence of integers $n_0, n_1, \ldots, n_l$ and $m_0, m_1, \ldots, m_{l-1}$ such that*

$$0 \leq n_0 < m_0 \leq n_1 < m_1 \leq \cdots \leq n_{l-1} < m_{l-1} \leq n_l \ .$$

The optimal dividend strategy π^* is given by the following:

$$\ell^{\pi^*}(y) = \begin{cases} 0 & \textit{for } 0 \le y \le n_0, \\ y - n_i & \textit{for } n_i < y < m_i,\ i = 0, 1, 2, \ldots, l-1, \\ 0 & \textit{for } m_{j-1} \le y \le n_j,\ j = 1, \ldots, l, \\ y - n_l & \textit{for } y \ge n_l\ . \end{cases}$$

The second type of extension to the original problem is to model X as a continuous time stochastic process. The most frequent and simple extensions involve either the Cramér-Lundberg process or a Brownian Motion with drift (see section 2.2 for the form of these two processes).

The latter was considered by Asmussen and Taksar [2]. The authors divide the problem into two cases: first, the case where the rate of dividend payout is restricted to some finite amount; and second, the case where dividend rates are unrestricted. In the unrestricted scenario the authors establish that a reflection strategy is optimal, and find explicit formulae for v_{π^*} and b^*. This result is not surprising given the findings of section 3.2 and the fact that Brownian motion is nothing other than the limiting process of a scaled symmetric random walk. To verify the claim, the authors first establish that $v_*(x)$ must satisfy the following HJB equation:

$$\max\left[\frac{1}{2}\sigma^2 v_*''(x) + \mu v_*'(x) - q v_*(x), 1 - v_*(x)\right] = 0,$$
$$v_*(0) = 0. \tag{3.17}$$

They further demonstrate the existence of a concave function f – for which they also find an explicit expression – and a barrier $b^* = \sup\{x : f'(x) > 1\}$ such that

$$f'(x) > 1 \qquad x < b^*\ \text{, and}$$
$$f'(x) = 1 \qquad x \ge b^*.$$

Using the principle of smooth fit they show f is a solution to (3.17), and establish an explicit formula for b^*. Finally, they demonstrate that $f(x) \ge v_\pi(x)$ for any $\pi \in \Pi$, and that for a particular π^* the inequality is actually an equality. This π^* has a cumulative dividend process of the form $L_t^{\pi^*} = \max\left[(\sup_{0\le s\le t} X_s - b^*), 0\right]$ which one recognizes as the cumulative dividend process of a reflection strategy. Thus a reflection strategy is established as optimal since $f(x) = v_{\pi^*}(x) \geq v_*(x)$.

An introductory treatment of the the Cramér-Lundberg case is given in section 6.4 of Bühlmann [6] in which one can find the following key result (attributed to Gerber [11]).

Theorem 16. *Let U_t be the net wealth of the company at time t. Under the condition that X is the Cramér-Lundberg process which has the form,*

$$X_t = x + ct - \sum_{i=1}^{N_t} \xi_n, \qquad t \geq 0, \tag{3.18}$$

there exists a number b^* such that for any initial capital $0 \leq x \leq b^*$ the following dividend policy is optimal:

- *If $U_t = b^*$ pay dividends at rate c.*
- *If $U_t < b^*$ pay no dividends.*

Note that the representation of the Cramér-Lundberg process in (3.18) is slightly different from the initial form presented in section 2.2. In (3.18) it is assumed that the ξ_n are positive random variables, the sum of which is subtracted from the positive drift term ct. Theorem (16) implies that the reflection strategy is optimal in the Cramér-Lundberg case when $x \leq b^*$. When $x > b^*$ the strategy is not necessarily optimal. Azcue and Muler [3] provide a counter-example to show that when ξ_n follows a gamma distribution with density $f(s) = s\mathrm{e}^{-s}$ the barrier strategy is not optimal for all $x > 0$. In this paper the authors show that a more general band strategy like the one proposed in theorem 15 is optimal. This result does not have the same intuitive appeal of the reflection strategy, and is often cited precisely because it is difficult to justify using financial arguments. The reflection strategy is, however, optimal when $\{\xi_n : n \geq 0\}$ is a sequence of i.i.d. exponential random variables. This suggests that the distribution of the claim amounts has influence over whether the reflection strategy is optimal or not. A more general version of this conclusion was reached by Loeffen [22] who showed that for spectrally negative Lévy processes, whose Lévy measures ν are completely monotone, optimality of the reflection strategy is guaranteed.

Theorem 17. *Let X be a spectrally negative Lévy process and suppose that the Lévy measure ν of X has a completely monotone density. Then a reflection strategy is optimal.*

Returning to the Cramér-Lundberg case where $\{\xi_n : n \geq 0\}$ is a sequence of i.i.d. exponential random variables one observes that the characteristic exponent is given by

$$\Psi(z) = -icz + \lambda \int_{-\infty}^{0} (1 - \mathrm{e}^{izs})\gamma \mathrm{e}^{\gamma s} ds \ . \tag{3.19}$$

From (3.19) it is clear that $\nu(s) = \lambda\gamma\mathrm{e}^{\gamma s}ds$ which indeed satisfies the conditions of theorem 17. That the reflection strategy is optimal for the Cramér-Lundberg case with exponential claim distribution has been known since long before theorem 17 was proven. Using techniques that are similar to the random walk case, but adapted for continuous time, one can demonstrate not only this result but also establish formulas for v_{π^*} and b^* which can be found in [6]. The next section will consider this model again under the condition that admissible strategies Π are restricted to those that limit the rate of dividend pay out. For those interested in solution techniques used to derive the unrestricted results, a proof will be given in the restricted case whose techniques are very similar.

The dividend strategies presented so far, whether of the barrier or band type, pay out all incoming wealth as soon as certain thresholds have been reached. Until now, no consideration has been given to the probability of ruin, just maximization of expected dividends. The following section will consider both more complicated processes, and a strategy that takes the probability of ruin into account.

3.4 The Refraction Strategy

In this section, it is assumed that X is a continuous time Markov process. Consider the optimal dividend problem where the admissible strategies Π are subject to the following restrictions:

1. L^π_t is absolutely continuous, that is, $L^\pi_t = \int_0^t \ell^\pi(s)ds$; and
2. $0 \leq \ell^\pi(t) \leq \delta < \infty$.

One notices that for many choices of X these restrictions immediately disqualify a reflection strategy. For example, in the case where X is a Brownian motion with positive drift it is not possible to describe $L^\pi_t = \max\left[\left(\sup_{0\leq s\leq t} X_s - b^*\right), 0\right]$ using a density because the set of points where L^π_t increases has Lebesgue measure 0. The impetus for restricting Π in such a way, is that $\mathbb{P}(\sigma^\pi < \infty) = 1$ when X

is a spectrally negative Lévy process and π is the reflection strategy. By choosing an appropriate δ one can insure that the infinite time horizon survival probability is positive for all $\pi \in \Pi$.

The purpose of this section is to introduce the refraction strategy and demonstrate that when X is a spectrally negative Lévy process and Π is restricted as above, a refraction strategy is optimal.

Definition 6. *A refraction strategy is a barrier strategy with barrier b that pays no dividends when $U_t^\pi < b$ and pays dividends at some rate $r < \infty$ when $U_t^\pi \geq b$.*

Note, that the terms "refraction" and "reflection" stem from the appearance of sample paths of the net wealth process controlled by the respective strategies. In Figure 3, X is a spectrally negative Lévy process, π_1 is a reflection strategy with barrier 10, and π_2 is a refraction strategy, also with barrier 10. Note how U is reflected under π_1 when it reaches 10, but only refracted under π_2.

The random walk example was used in the previous section to gain insight into the proof techniques needed to establish optimality of the reflection strategy and calculate the value function. In the continuous time setting the Cramér-Lundberg case with exponential claim distribution will serve the same purpose. It will be shown that when Π is restricted as above, a refraction strategy with rate δ is optimal. A full proof of this is provided in [13]. Unless otherwise stated, the reader may assume all theorems, lemmas, propositions, facts and associated proofs in the next section are taken from [13].

3.4.1 Cramér-Lundberg Process

Theorem 18. *Assume that X is the Cramér-Lundberg process with exponential claim density $f(s) = \gamma e^{-\gamma s}$ and $X_t = x + ct - \sum_{n=1}^{N_t} \xi_n$. Further assume, Π is restricted as above with $\delta < c$ so that $dL_t^\pi \leq \delta dt$ for all $\pi \in \Pi$. Then, there exists a barrier b^* such that a refraction strategy π_{b^*} with rate δ is optimal. Further, the value function has the form*

$$v_{\pi_{b^*}}(x) = \begin{cases} \frac{\zeta_2}{\gamma} \frac{\delta}{q} \frac{(\gamma+\Phi)e^{\Phi x} - (\gamma-\zeta_1)e^{-\zeta_1 x}}{(\zeta_2+\Phi)e^{\Phi b^*} - (\zeta_2-\zeta_1)e^{-\zeta_1 b^*}} & 0 < x \leq b^* \\ \frac{\delta}{q}(1 - e^{-\zeta_2(x-b^*)}) + v_{\pi_{b^*}}(b^*)e^{-\zeta_2(x-b^*)} & x \geq b^*, \end{cases} \tag{3.20}$$

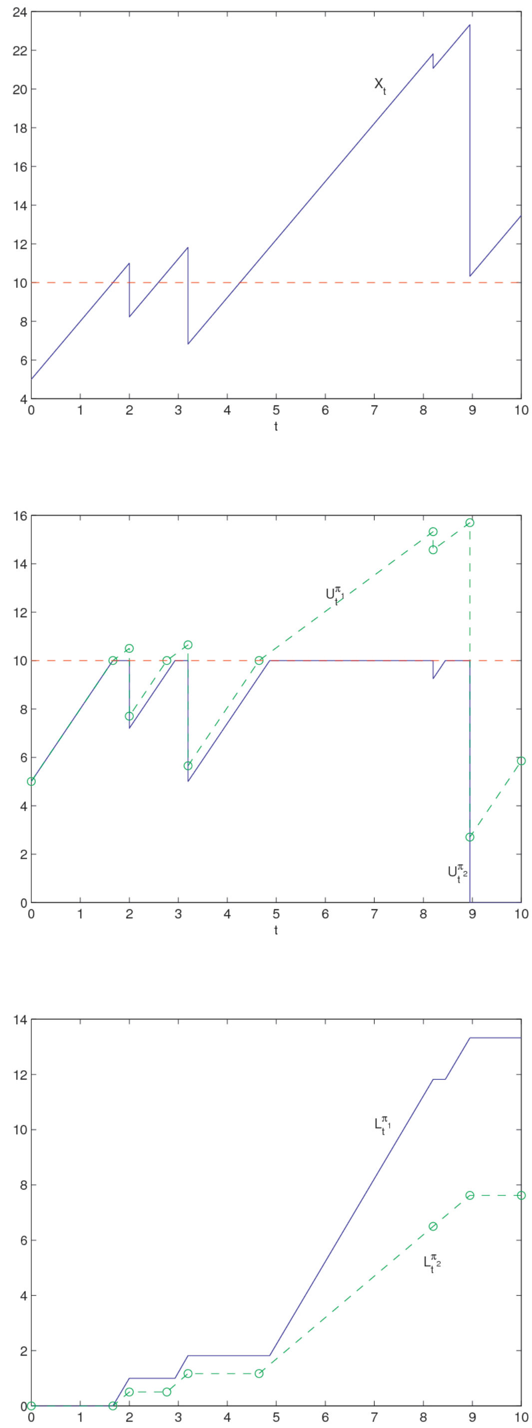


Figure 3: A comparison of the net wealth process (middle) and cumulative dividend process (bottom) for the refraction (π_1) and reflection (π_2) strategies, both with optimal barrier 10, applied to the same path of the gross wealth process (top).

where $\Phi > 0$ *and* $-\zeta_1 < 0$ *are the roots of the equation*

$$cz^2 + (\gamma c - \lambda - q)z - \gamma q = 0 \text{ , and}$$

$-\zeta_2 < 0$ *is the negative root of the equation*

$$(c-\delta)z^2 + (\gamma(c-\delta) - \lambda - q)z - \gamma q = 0.$$

Proof. As in the case of the random walk, establishing that the refraction strategy is optimal involves a rather lengthy proof. The first lemma will seem familiar from the example of the random walk.

Lemma 19. $v_*(x) = \sup_{\pi\in\Pi} v_\pi(x)$ *must satisfy the following HJB equation:*

$$\max_{0\le r\le\delta}[r + (c-r)v_*'(x)] - (\lambda+q)v_*(x) + \lambda\int_0^x v_*(x-y)\gamma e^{-\gamma y}dy = 0, \qquad x \ge 0 \ . \tag{3.21}$$

Proof. To verify this proposition consider first the "small" time interval between 0 and dt. During this time suppose one pays dividends at rate r, and then applies v_* to the net wealth at time dt. Now, one can condition on the event that a claim occurs during dt. The value under this strategy is

$$\begin{aligned}
v_r(x) &= rdt + \mathrm{e}^{-qdt}\left[(1-\lambda dt)v_*(x+(c-r)dt) + \lambda dt\int_0^x v_*(x-y)\gamma\mathrm{e}^{-\gamma y}dy\right] + o(dt)\\
&= rdt + (1-qdt)\left[(1-\lambda dt)v_*(x+(c-r)dt) + \lambda dt\int_0^x v_*(x-y)\gamma\mathrm{e}^{-\gamma y}dy\right] + o(dt)\\
&= rdt + \Bigg[v_*(x+(c-r)dt) - qdtv_*(x+(c-r)dt)\\
&\quad -\lambda dtv_*(x+(c-r)dt) + \lambda dt\int_0^x v_*(x-y)\gamma\mathrm{e}^{-\gamma y}dy\Bigg] + o(dt)\\
&= v_*(x) + \Bigg[r + (c-r)\left(\frac{v_*(x+(c-r)dt) - v_*(x)}{(c-r)dt}\right)\\
&\quad -(q+\lambda)v_*(x+(c-r)dt) + \lambda\int_0^x v_*(x-y)\gamma\mathrm{e}^{-\gamma y}dy\Bigg]dt + o(dt) \ .
\end{aligned}$$

Note that $v_r(x)$ becomes optimal if one maximizes the above expression with respect to $r \in [0,\delta]$. However, since $v_*(x)$ already appears in the last equality the term in brackets must be equal to 0 when maximized with respect to r. Thus one arrives at (3.21). □

It becomes apparent that the optimal rate at $t = 0$ is that one that maximizes the expression $r(1 - v'(x))$. Let U_t be the net wealth at time t. If one considers that the preceding analysis is valid for any time t one must conclude that the optimal dividend rate at time t is

$$\begin{aligned} &0 \text{ if } v_*'(U_t) > 1, \\ &\delta \text{ if } v_*'(U_t) < 1. \end{aligned} \tag{3.22}$$

It has been shown that $v_*(x)$ must necessarily satisfy HJB. However, one can further demonstrate that if π^* is such that v_{π^*} satisfies HJB, then $v_{\pi^*} = v_*(x)$.

Proposition 20 (Proof modified from [13]). *If π^* is such that v_{π^*} satisfies HJB, then $v_{\pi^*} = v_*(x)$.*

Proof. Suppose v_{π^*} satisfies HJB, and let π be another strategy with rate $\ell^\pi(t)$. Now consider the process $Y = \{Y_t : t \geq 0\}$ where,

$$Y_t = \mathrm{e}^{-qt} v_{\pi^*}(U_t^{\pi^*}) + \int_0^t \mathrm{e}^{-qs} \ell^\pi(s) ds \ .$$

Let $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ be the filtration to which X, π, and π^* are adapted. Then,

$$\begin{aligned}
\mathbb{E}\left[Y_{t+\Delta t} \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\mathrm{e}^{-q(t+\Delta t)} v_{\pi^*}(U_{t+\Delta t}^{\pi^*}) + \int_0^{t+\Delta t} \mathrm{e}^{-qs} \ell^\pi(s) ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-qt}\mathrm{e}^{-\Delta t}\mathbb{E}\left[v_{\pi^*}(U_{t+\Delta t}^{\pi^*}) \,\middle|\, \mathcal{F}_t\right] + \int_0^t \mathrm{e}^{-qs}\ell^\pi(s)ds + \mathbb{E}\left[\int_t^{t+\Delta t} \mathrm{e}^{-qs}\ell^\pi(s)ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-qt}(1 - q\Delta t)\Bigg[(1 - \lambda\Delta t)v_{\pi^*}(U_t^{\pi^*} + (c - \ell^{\pi^*}(t))\Delta t) \\
&\quad + \lambda\Delta t \int_0^{U_t^{\pi^*}} v_{\pi^*}(U_t^{\pi^*} - y)\gamma \mathrm{e}^{-\gamma y} dy\Bigg] + \int_0^t \mathrm{e}^{-qs}\ell^\pi(s)ds + \mathrm{e}^{-qt}\ell^\pi(t)\Delta t + o(\Delta t) \\
&= \mathrm{e}^{-qt} v_{\pi^*}(U_t^{\pi^*}) + \int_0^t \mathrm{e}^{-qs}\ell^\pi(s)ds \\
&\quad + \mathrm{e}^{-qt}\Delta t \Bigg[\ell^\pi(t) + (c - \ell^\pi(t))\left(\frac{v_{\pi^*}(U_t^{\pi^*} + (c - \ell^\pi(t))\Delta t) - v_{\pi^*}(U_t^{\pi^*})}{(c - \ell^\pi(t))\Delta t}\right) \\
&\quad - (q + \lambda)v_{\pi^*}(U_t^{\pi^*} + (c - \ell^\pi(t))\Delta t) + \lambda \int_0^{U_t^{\pi^*}} v_{\pi^*}(U_t^{\pi^*} - y)\gamma \mathrm{e}^{-\gamma y} dy\Bigg] + o(\Delta t) \\
&\leq \mathrm{e}^{-qt} v_{\pi^*}(U_t^{\pi^*}) + \int_0^t \mathrm{e}^{-qs}\ell^\pi(s)ds = Y_t \ .
\end{aligned}$$

The last inequality arises because v_{π^*} satisfies HJB so that the term in brackets must be less than or equal to 0. This shows that Y is a positive supermartingale so that,

$$\mathbb{E}_x\left[Y_\infty \mid \mathcal{F}_0\right] = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-qs}\ell^\pi(s)ds \,\middle|\, X_0 = x\right] = v_\pi(x) \leq Y_0 = v_{\pi^*}(x) \ .$$

Since π was arbitrary the preceding arguments establish that π^* is optimal. □

This analysis shows that the refraction strategy will be optimal if and only if it satisfies HJB or equivalently condition (3.22). Thus if π_{b^*} is indeed optimal $v_{\pi_{b^*}}$ must satisfy

$$cv'_{\pi_{b^*}}(x) - (\lambda+q)v_{\pi_{b^*}}(x) + \lambda\int_0^x v_{\pi_{b^*}}(x-y)\gamma\mathrm{e}^{-\gamma y}dy = 0, \qquad 0 < x < b^*, \tag{3.23}$$

$$\delta + (c-\delta)v'_{\pi_{b^*}}(x) - (\lambda+q)v_{\pi_{b^*}}(x) + \lambda\int_0^x v_{\pi_{b^*}}(x-y)\gamma\mathrm{e}^{-\gamma y}dy = 0, \qquad x > b^*. \tag{3.24}$$

Further, $v_{\pi_{b^*}}$ is bounded above since

$$\lim_{x\to\infty} v_{\pi_{b^*}}(x) = \frac{\delta}{q} \ . \tag{3.25}$$

This follows from the fact that none of the restricted strategies can exceed a perpetual payment of dividends at rate δ. One can demonstrate that $v_{\pi_{b^*}}(x)$ is continuous in x when $x = b^*$, but $v'_{\pi_{b^*}}(x)$ does not necessarily share this property. However, from (3.23) and (3.24) one can establish the equality

$$cv'_{\pi_{b^*}}(b^*-) = (c-\delta)v'_{\pi_{b^*}}(b^*+) + \delta \ . \tag{3.26}$$

Additionally, the equation

$$ch'(x) - (\lambda+q)h(x) + \lambda\int_0^x h(x-y)\gamma\mathrm{e}^{-\gamma y}dy = 0, \qquad 0 < x \ . \tag{3.27}$$

has up to a constant factor α a unique solution $h(x)$. From this and (3.23) it follows that

$$v_{\pi_{b^*}}(x) = \alpha h(x), \qquad 0 \leq x \leq b^* \ . \tag{3.28}$$

Using the properties of convolution, and the form of the exponential density one can convert (3.27) into a homogeneous differential equation of the form,

$$ch''(x) + (\gamma c - \lambda - q)h'(x) - \gamma q h(x) = 0, \qquad 0 < x \ , \tag{3.29}$$

which can be solved to show that $h(x)$ is proportional to $(\gamma + \Phi)\mathrm{e}^{\Phi x} - (\gamma - \zeta_1)\mathrm{e}^{-\zeta_1 x}$. Here $\Phi > 0$ and $-\zeta_1 < 0$ are the roots of the characteristic equation for (3.29). This result, together with (3.28) gives

$$v_{\pi_{b^*}}(x) = \alpha((\gamma + \Phi)\mathrm{e}^{\Phi x} - (\gamma - \zeta_1)\mathrm{e}^{-\zeta_1 x}), \qquad 0 < x \leq b^* \ . \tag{3.30}$$

Then in the case $\delta = c$, one can employ(3.26), (3.28), and (3.30) to find

$$\alpha = \frac{1}{\Phi(\gamma + \Phi)\mathrm{e}^{\Phi b^*} + \zeta_1(\gamma - \zeta_1)\mathrm{e}^{-\zeta_1 b^*}} \ , \tag{3.31}$$

which, together with (3.30) gives the value of the reflection strategy with barrier b^* for $0 < x \leq b^*$ (see discussion following theorem 17).

Turning now to the case where $x > b^*$, one can apply the same techniques as for the $x \leq b^*$ to alter (3.24) to the differential equation

$$(c - \delta)v''_{\pi_{b^*}}(x) + (\gamma(c - \delta) - \lambda - q)v'_{\pi_{b^*}}(x) - \gamma q v_{\pi_{b^*}}(x) + \gamma\delta = 0, \qquad x > b^* \ . \tag{3.32}$$

This has $\frac{\delta}{q}$ as a particular solution, and general solution

$$v_{\pi_{b^*}}(x) = \frac{\delta}{q} + D\mathrm{e}^{-\zeta_2 x}, \qquad x \geq b^* \ . \tag{3.33}$$

In (3.33) D is a constant to be determined and $-\zeta_2 < 0$ is the negative root of the characteristic equation. To find the value of the refraction strategy with barrier b^* and rate δ all that remains is to find values for the as yet unknown α and D. This is accomplished using the continuity of $v_{\pi_{b^*}}(x)$ at b^*, the properties of the convolution, and the form of the exponential density to yield

$$v_{\pi_{b^*}}(x) = \begin{cases} \frac{\zeta_2}{\gamma}\frac{\delta}{q}\frac{(\gamma+\Phi)e^{\Phi x}-(\gamma-\zeta_1)e^{-\zeta_1 x}}{(\zeta_2+\Phi)e^{\Phi b^*}-(\zeta_2-\zeta_1)e^{-\zeta_1 b^*}} & 0 < x \le b^* \quad \text{, and} \\ \frac{\delta}{q}(1-e^{-\zeta_2(x-b^*)}) + v_{\pi_{b^*}}(b^*)e^{-\zeta_2(x-b^*)} & x \ge b^*. \end{cases} \tag{3.34}$$

Note that $v_{\pi_0}(0) = \frac{\zeta_2}{\gamma}\frac{\delta}{q}$. It can be verified that $0 < \frac{\zeta_2}{\gamma} < 1$ so that $0 < v_{\pi_0}(0) < \frac{\delta}{q}$

Now it is necessary to determine whether (3.34) satisfies HJB or equivalently (3.22). Thus π_{b^*} will be optimal if the following conditions hold:

$$v'_{\pi_{b^*}}(x) > 1 \text{ for } x < b^* \tag{3.35}$$

$$v'_{\pi_{b^*}}(x) < 1 \text{ for } x > b^* \ . \tag{3.36}$$

Taking the derivative of (3.34) when $x > b^*$ gives

$$v'_{\pi_{b^*}}(x) = \zeta_2\left(\frac{\delta}{q} - v_{\pi_{b^*}}(b^*)\right)e^{\zeta_2(x-b^*)}, \qquad x > b^* \tag{3.37}$$

Suppose that

$$\begin{aligned} v'_{\pi_0}(x) &= \zeta_2\left(\frac{\delta}{q} - v_{\pi_0}(0)\right) \\ &= \zeta_2\frac{\delta}{q}\left(1 - \frac{\zeta_2}{\gamma}\right) \le 1 \ . \end{aligned}$$

Then, the refraction strategy with $b^* = 0$, satisfies (3.36) and is thus optimal. Alternatively, suppose that

$$v'_{\pi_0}(x) = \zeta_2\frac{\delta}{q}\left(1 - \frac{\zeta_2}{\gamma}\right) > 1 \ . \tag{3.38}$$

Equation (3.26) shows that $v'_{\pi_{b^*}}(b^*-)$ is the weighted average of $v'_{\pi_{b^*}}(b^*+)$ and 1. Thus the quantities $v'_{\pi_{b^*}}(b^*-)$ and $v'_{\pi_{b^*}}(b^*+)$ are both less than 1, both greater than 1, or both equal to 1. From this, (3.35), and (3.36) one can conclude that b^* must satisfy

$$v'_{\pi_{b^*}}(b^*-) = 1 = v'_{\pi_{b^*}}(b^*+) \tag{3.39}$$

Together with (3.37) this implies that

$$\zeta_2\left(\frac{\delta}{q} - v_{\pi_{b^*}}(b^*)\right) = 1 \ , \tag{3.40}$$

so that b^* is the solution of the equation

$$v_{\pi_{b^*}}(b^*) = \frac{\delta}{q} - \frac{1}{\zeta_2} \ . \tag{3.41}$$

From (3.34) one can then obtain

$$v_{\pi_{b^*}}(x) = \frac{\delta}{q} - \frac{1}{\zeta_2}\mathrm{e}^{-\zeta_2(x-b^*)}, \qquad x \geq b^*. \tag{3.42}$$

With this information, all that remains is to check that $v'_{\pi_{b^*}}(x)$ satisfies (3.35) and (3.36). It is quite clear from (3.42) that (3.36) is satisfied since

$$v'_{\pi_{b^*}}(x) = \mathrm{e}^{-\zeta_2(x-b^*)} < 1, \qquad x > b^*. \tag{3.43}$$

A slightly more complicated argument shows that $v'_{\pi_{b^*}}(x) > 1$ when $0 \leq x < b^*$ which completes the proof of optimality. To find b^* one can solve either (3.41), (3.39) or one can find the value of b^* that maximizes (3.34). This approach yields

$$b^* = \frac{1}{\Phi + \zeta_1}\ln\left(\frac{\zeta_1^2 - \zeta_1\zeta_2}{\Phi^2 + \zeta_2\Phi}\right). \tag{3.44}$$

Remark The form of the expression (3.44) indicates that $b^* = 0$ when $(\zeta_1^2 - \zeta_1\zeta_2)/(\Phi^2 + \zeta_2\Phi) = 1$. The authors of [13] rewrite this expression in terms of the original parameters as follows:

$$\delta = q\frac{\gamma c^2}{(\lambda + q)(\gamma c - \lambda - \delta)} \tag{3.45}$$

From here they conclude that:

1. b^* is 0 if the denominator of (3.45) is negative or 0.
2. b^* is 0 if the denominator of (3.45) is positive but δ is less than or equal to the expression on the right hand side.
3. b^* is positive if δ is greater than the expression on the right hand side.

□

3.4.2 Spectrally Negative Lévy Processes

The primary object of focus of this thesis are the results found in [19]. With slightly more restrictions on Π and δ than in the Cramér-Lundberg scenario Kyprianou, Loeffen, and Pérez show that when X is a spectrally negative Lévy process that has a Lévy measure with a completely monotone density a refraction strategy π_{b^*} with maximal rate δ is optimal.

Before proceeding to the statement of the main theorem, recall from section 2.3 that

$$a^* = \sup\left\{a \geq 0 : W^{(q)'}(a) \leq W^{(q)'}(x) \text{ for all } x \geq 0\right\} .$$

Also, recall that when ν has a completely monotone density $0 < a^* < \infty$ is the unique point at which the strictly convex function $W^{(q)'}$ attains its minimum. In this section ψ will continue to denote the Laplace exponent of a spectrally negative Lévy process and for $q \geq 0$, $\Phi(q)$ will denote the largest positive root of the equation $\psi(z) = q$. Additionally, $\mathbb{W}^{(q)}$ will represent the scale function of the linearly perturbed process $Y = \{Y_t = X_t - \delta t : t \geq 0\}$ where $0 < \delta < \mathbb{E}[X_1]$, and $\varphi(q) = \sup\{z \geq 0 : \psi(z) - \delta z = q\}$. Note, under these conditions Y remains a spectrally negative process. If $\psi(z)$ denotes the Laplace exponent of the process X it is not difficult to determine that $\psi(z) - \delta z$ is the Laplace exponent of Y and $\varphi(q)$ plays the role of $\Phi(q)$ for Y. Given the properties of Laplace exponents for spectrally negative Lévy processes found in section 2.3 and the fact that $\psi(z) > \psi(z) - \delta z$ for all z, it is clear that $\varphi(q) > \Phi(q)$.

Theorem 21 (Theorem 1, and Lemma 4, [19])**.** *Assume the following conditions hold:*

1. *X is a spectrally negative Lévy process with Lévy triplet given by (a, σ, ν) such that ν has a completely monotone density. Recall that X has Laplace exponent:*

$$\psi(z) = -az + \frac{1}{2}\sigma^2 z^2 + \int_{(-\infty,0)} (e^{zx} - 1 - zx\mathbf{1}_{\{-1\le x<0\}})\nu(dx).$$

2. *$\delta > 0$ is a rate such that $\delta < d = -a + -\int_{(-1,0)} x\nu(dx)$ when X has paths of bounded variation.*

3. *Π consists of strategies of the form $\pi = \{L_t^\pi : t \ge 0\}$ which denote absolutely continuous, non-negative, non-decreasing, left-continuous processes adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t : t \ge 0\}$ of X such that*

$$L_t^\pi = \int_0^t \ell^\pi(s)ds \ ,$$

and for $t \ge 0$, $\ell^\pi(t)$ satisfies

$$0 \le \ell^\pi(t) \le \delta \ .$$

Then, an optimal dividend strategy is given by a refraction strategy with rate δ. The value function has the form

$$v_{\pi_{b^*}}(x) = -\delta \int_0^{x-b^*} \mathbb{W}^{(q)}(y)dy + \frac{W^{(q)}(x) + \delta \int_{b^*}^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy}{h(b^*)}, \qquad x \ge 0, \tag{3.46}$$

where $h(b^*)$ is given by

$$h(b^*) = \varphi(q)e^{\varphi(q)b^*} \int_{b^*}^\infty e^{-\varphi(q)y}W^{(q)\prime}(y)dy, \tag{3.47}$$

and the optimal barrier $b^* \in [0, a^*)$ is the unique point at which h attains its minimum. Moreover, $b^* > 0$ if and only if one of the following three cases hold:

(i) $\sigma > 0$ *and* $\varphi(q) < 2\delta/\sigma^2$

(ii) $\sigma = 0$, $\nu(-\infty, 0) < \infty$, *and* $\varphi(q) < \delta(\nu(-\infty, 0) + q)/d(d-q)$

(iii) $\sigma = 0$ *and* $\nu(-\infty, 0) = \infty$.

Proof (sketch). As may be expected, the proof of this general result is considerably more technical and difficult than the one presented in the previous examples. However, once (3.46) has been established as a candidate value function, and cases (i) - (iii) have been deduced for b^*, the pattern of proof is very similar to that of the proofs already presented in the examples. That is, one first establishes the HJB equation, demonstrates that if for some π the value function v_π satisfies the equation then π must be optimal, establishes an equivalent condition to the HJB equation almost identical to (3.35) and (3.36), and finally uses this last condition to demonstrate that $v_{\pi_{b^*}}$ satisfies the HJB equation. Selected details of the proof follow. For the entire proof and associated background see [19] and [21].

First, it is necessary to establish (3.46) as a candidate function. Its derivation stems from the properties of so-called refracted Lévy processes. A refracted Lévy process U is a unique strong solution to the stochastic differential equation

$$dU_t = -\delta \mathbf{1}_{\{U_t > b\}} + dX_t \ ,$$

from which it follows that $U^{\pi_{b^*}}$ is a refracted processes. General results for the case when X is a spectrally negative process are gathered in [21]. Among these results is the derivation of (3.46). A proof is omitted as the underlying theory is fairly advanced, and would require presenting a significant amount background material.

If one accepts the candidate function as valid, one can begin to prove the statements about the barrier b^*. This will be done through a sequence of propositions.

Proposition 22. *b^* is finite.*

Proof.

$$\begin{aligned}
h(b) &= \varphi(q)\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)\prime}(y)dy \\
&= \varphi(q)\int_0^\infty \mathrm{e}^{-\varphi(q)u}W^{(q)\prime}(u+b)du \\
&= \varphi(q)\left[\mathrm{e}^{\Phi(q)b}\mathrm{e}^{(\Phi(q)-\varphi(q))u}W_{\Phi(q)}(u+b)\Big|_0^\infty + \varphi(q)\int_0^\infty \mathrm{e}^{-\varphi(q)u}W^{(q)}(u+b)du\right] \\
&= \varphi(q)^2\mathrm{e}^{\Phi(q)b}\int_0^\infty \mathrm{e}^{-\varphi(q)u}\left(\mathrm{e}^{\Phi(q)u}W_{\Phi(q)}(u+b) - W_{\Phi(q)}(b)\right)du \\
&\geq \varphi(q)^2\mathrm{e}^{\Phi(q)b}W_{\Phi(q)}(b)\int_0^\infty \mathrm{e}^{-\varphi(q)u}\left(\mathrm{e}^{\Phi(q)u} - 1\right)du
\end{aligned}$$

$$= W^{(q)}(b)\frac{\Phi(q)\varphi(q)}{\varphi(q) - \Phi(q)} \; .$$

The third equality above is the result of integration by parts. Additionally, the fact that $\Phi(q) < \varphi(q)$ has been used several times in the above calculations. Since $W^{(q)}(\infty) = \infty$ it follows that $\lim_{b\to\infty} h(b) = \infty$ which shows that $b^* < \infty$ as desired. □

Proposition 23. *b^* is the unique point at which h attains its minimum.*

Proof. One can differentiate h with respect to b to obtain

$$h'(b) = \varphi(q)(h(b) - W^{(q)'}(b)).$$

This shows that $h'(b)$ is negative when $h(b)$ is less than $W^{(q)'}(b)$ and positive when $h(b)$ is greater than $W^{(q)'}(b)$. From the assumptions it is known that $W^{(q)'}$ is strictly convex with $W^{(q)'}(\infty) = \infty$. Since $h(\infty) = \infty$ this shows that either $h(b) > W^{(q)'}(b)$ for all $b \in [0, \infty)$ in which case h is strictly increasing on $[0, \infty)$ and $b^* = 0$, or there is a unique point $b^* \in [0, \infty)$ at which $h(b^*) = W^{(q)'}(b^*)$, so that for $b < b^*$, $h(b) < W^{(q)'}(b^*)$ and for $b > b^*$, $h(b) > W^{(q)'}(b^*)$. In either case b^* is the unique point at which h attains its minimum. See figure 4 for a graphical representation of this argument.

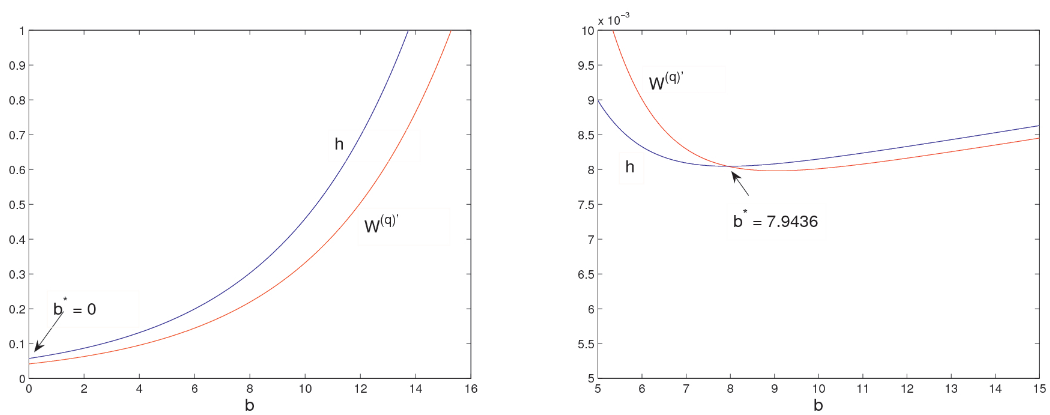


Figure 4: $W^{(q)'}$ and h for two different Cramér-Lundberg processes. The left-hand side depicts the case $h(0) \geq W^{(q)'}(0)$ and $b^* = 0$. The right-hand side shows the case $h(0) < W^{(q)'}(0)$ and demonstrates that b^* is the unique point in $[0, \infty)$ at which h and $W^{(q)'}$ are equal.

□

Proposition 24. $b^* \in [0, a^*)$

Proof. To show that $b^* < a^*$ one can proceed by contradiction. First assume that $b^* > a^*$. Since $W^{(q)\prime}(b)$ is convex, and a^* is the global minimum of $W^{(q)\prime}$, $W^{(q)\prime\prime}(b) > 0$ for all $b \geq b^*$. Thus there exists $L > 0$ and $\varepsilon > 0$ such that for $0 < b - b^* < \varepsilon$

$$L < \frac{W^{(q)\prime}(b) - W^{(q)\prime}(b^*)}{b - b^*} \text{ , and}$$

$$0 < \frac{h(b) - h(b^*)}{b - b^*} < \frac{L}{2}.$$

Thus,

$$\frac{W^{(q)\prime}(b) - W^{(q)\prime}(b^*)}{b - b^*} - \frac{h(b) - h(b^*)}{b - b^*} > 0 \ ,$$

from which it follows that

$$W^{(q)\prime}(b) > h(b) \ ,$$

for all $b \in (b^*, b^* + \varepsilon)$. This statement is a contradiction to the conclusions of proposition 23 which showed $h(b) > W^{(q)\prime}(b)$ when $b > b^*$. Now assume that $b^* = a^*$. Since a^* is the unique minimum of $W^{(q)\prime}$, it follows that $W^{(q)\prime}(u + a^*) > W^{(q)\prime}(a^*)$ for all $u \in (0, \infty)$. Thus,

$$\begin{aligned} h(a^*) &= \varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q)u} W^{(q)\prime}(u + a^*) du \\ &> \varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q)u} W^{(q)\prime}(a^*) du \\ &= W^{(q)\prime}(a^*) \ . \end{aligned}$$

Since it was shown in proposition 23 that $h(b^*)$ must equal $W^{(q)\prime}(b^*)$ the preceding calculation identifies a contradiction. Thus, $b^* \in [0, a^*)$ as claimed. ☐

Proposition 25. $b^* > 0$ *if and only if one of (i) - (iii) holds.*

Proof. From the arguments made in proposition 23 it is clear that $b^* > 0$ if and only if $h(0) < W^{(q)'}(0+)$. One can evaluate $h(0)$ as follows:

$$
\begin{aligned}
h(0) &= \varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q)u} W^{(q)'}(u) du \\
&= \varphi(q) \left[\mathrm{e}^{(\Phi(q)-\varphi(q))u} W_{\Phi(q)}(u) \Big|_0^\infty + \varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q)u} W^{(q)}(u) du \right] \\
&= \varphi(q) \left[\frac{\varphi(q)}{\psi(\varphi(q)) - q} - W^{(q)}(0) \right] \\
&= \varphi(q) \left[\frac{1}{\delta} - W^{(q)}(0) \right] \ . \qquad (3.48)
\end{aligned}
$$

Above, the second equality results from integration by parts, the third from the definition of $W^{(q)}$ and the fact that $\varphi(q) > \Phi(q)$, and the fourth from the definition of $\varphi(q)$. Now one can refer to lemma 2 to establish the enumerated properties of b^*. For example, if $\sigma > 0$, X has paths of unbounded variation so that $W^{(q)}(0) = 0$ and $W^{(q)'}(0) = 2/\sigma^2$. Condition (i) follows easily when these results are combined with (3.48) and the remaining two conditions are obtained similarly. □

To complete the proof of the remainder of the theorem, it is necessary to show that $v_{\pi_{b^*}}$ is indeed optimal. For this purpose, one needs to define the concept of a "sufficiently smooth" function of a spectrally negative Lévy process X. A function f is *sufficiently smooth* if it is continuously differentiable on $(0, \infty)$ when X has bounded variation, and twice continuously differentiable on $(0, \infty)$ when X has unbounded variation. Let Γ be the operator acting on sufficiently smooth functions which is defined as

$$
\Gamma f(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(0,\infty)} f(x-z) - f(x) + f'(x) z \mathbf{1}_{\{0 \le z \le 1\}} \nu(dz) \ .
$$

Then the HJB inequality for this problem can be expressed as follows.

Lemma 26 (Hamilton-Jacobi-Bellman Inequality)**.** *Suppose π is an admissible dividend strategy such that v_π is sufficiently smooth on $(0, \infty)$, right-continuous at zero, and for all $x > 0$*

$$
\sup_{0 \le r \le \delta} \left[\Gamma v_\pi(x) - q v_\pi(x) - r v'_\pi(x) + r \right] \le 0 \ . \qquad (3.49)
$$

Then $v_\pi(x) = v_*(x)$ for all $x \geq 0$ and hence π is an optimal strategy.

A proof of lemma 26 will not be given here as the arguments needed are rather technical and the appearance of a HJB condition is not surprising given earlier examples.

Showing that lemma 26 applies to $v_{\pi_{b^*}}$ – that is, proving that $v_{\pi_{b^*}}$ is sufficiently smooth – is simply a matter of differentiating twice and demonstrating that both the first and second derivatives are continuous. This is not difficult, since assumptions ensure that both $W^{(q)}$ and $\mathbb{W}^{(q)}$ are infinitely continuously differentiable. Showing that $v_{\pi_{b^*}}$ satisfies (3.49) is considerably more difficult. To do so, one must first demonstrate that $v_{\pi_{b^*}}$ satisfies (3.49) if and only if

$$\begin{cases} v'_{\pi_{b^*}}(x) \geq 1 & \text{if } 0 < x \leq b^* \\ v'_{\pi_{b^*}}(x) \leq 1 & \text{if } x > b^*. \end{cases} \tag{3.50}$$

Considering earlier cases – the Brownian motion example in section 3.3, or the Cramér-Lundberg example of section 3.4.1 – this condition seems familiar, even if it is slightly different in each case.

To conclude the proof, one must show that $v_{\pi_{b^*}}$ satisfies (3.50). A proof is included for the case when $b^* = 0$.

Proposition 27. *When $b^* = 0$, the value function v_{π_0} satisfies (3.50) so that π_0 is the optimal dividend strategy.*

Proof. First, assume that $b^* = 0$ or equivalently that either

(i) $\sigma > 0$ and $\varphi(q) \geq 2\delta/\sigma^2$, or

(ii) $\sigma = 0$, $\nu(-\infty, 0) < \infty$, and $\varphi(q) \geq \delta(\nu(-\infty, 0) + q)/\mathrm{d}(\mathrm{d} - q)$.

Then, for $x > 0$ one can use (3.46) to show that

$$v_{\pi_0}(x) = -\delta \left(\int_0^x \mathbb{W}^{(q)}(y)dy - \frac{1}{\varphi(q)} \mathbb{W}^{(q)}(x) \right). \tag{3.51}$$

Deriving equation (3.51) is not straightforward. Let $f(x) = \int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy$. Then,

$$
\begin{aligned}
F(s) = \int_0^\infty \mathrm{e}^{-sx} f(x)dx &= \int_0^\infty \mathrm{e}^{-sx}\left(\int_0^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy\right)dx \\
&= \int_0^\infty W^{(q)\prime}(y)\int_y^\infty \mathrm{e}^{-sx}\mathbb{W}^{(q)}(x-y)dxdy \\
&= \int_0^\infty \mathrm{e}^{-sy}W^{(q)\prime}(y)\int_0^\infty \mathrm{e}^{-su}\mathbb{W}^{(q)}(u)dudy \\
&= \int_0^\infty \mathrm{e}^{-sy}W^{(q)\prime}(y)dy\left(\frac{1}{\psi(s)-\delta s-q}\right) \\
&= \left(-W^{(q)}(0)+\frac{s}{\psi(s)-q}\right)\left(\frac{1}{\psi(s)-\delta s-q}\right) \\
&= \frac{-W^{(q)}(0)}{\psi(s)-\delta s-q} - \frac{1}{\delta}\left(\frac{1}{\psi(s)-q}-\frac{1}{\psi(s)-\delta s-q}\right),
\end{aligned}
\tag{3.52}
$$

and

$$
f(x) = \mathcal{L}^{-1}\left\{F(s)\right\}(x) = -W^{(q)}(0)\mathbb{W}^{(q)}(x) - \frac{1}{\delta}W^{(q)}(x) + \frac{1}{\delta}\mathbb{W}^{(q)}(x). \tag{3.53}
$$

The third equality is the result of a change of variables, and the sixth is the result of a partial fraction decomposition. With (3.46),(3.48), and (3.53) one can verify the validity of (3.51), and calculate

$$
v_{\pi_0}'(x) = -\delta\left(\mathbb{W}^{(q)}(x) - \frac{1}{\varphi(q)}\mathbb{W}^{(q)\prime}(x)\right). \tag{3.54}
$$

Now one can make use of the decomposition of the scale function given in lemma 4 to write

$$
\begin{aligned}
v_{\pi_0}'(x) &= -\delta\left(\frac{\mathrm{e}^{\varphi(q)x}}{\psi'(\varphi(q))} - f(x) - \frac{1}{\varphi(q)}\left(\frac{\varphi(q)\mathrm{e}^{\varphi(q)x}}{\psi'(\varphi(q))} - f'(x)\right)\right) \\
&= -\delta\left(\frac{f'(x)}{\varphi(q)} - f(x)\right).
\end{aligned}
\tag{3.55}
$$

Differentiating (3.55) gives

$$
v_{\pi_0}''(x) = -\delta\left(\frac{f''(x)}{\varphi(q)} - f'(x)\right). \tag{3.56}
$$

Since f is a completely monotone function and $\varphi(q) > 0$, $v_{\pi_0}''(x) < 0$ for $x > 0$. Therefore, v_{π_0}'

is decreasing on $(0, \infty)$. If one can show that $v'_{\pi_0}(0+) \le 1$ then $v_{\pi_{b^*}}$ satisfies (3.50) as desired. Requiring that $v'_{\pi_0}(0+) \le 1$ is equivalent to requiring that

$$\frac{\delta \mathbb{W}^{(q)\prime}(0+)}{1+\delta \mathbb{W}^{(q)}(0)} \le \varphi(q). \tag{3.57}$$

All that remains is to check that (3.57) is satisfied for the cases for which $b^* = 0$. This is easily accomplished by using the results of lemma 2, which demonstrates that π_{b^*} is optimal when $b^* = 0$. □

A more complicated argument establishes the same result for the case $b^* > 0$ thereby verifying π_{b^*} as the optimal dividend strategy.

□

3.4.3 Examples

Theorem 21 allows one to quickly obtain results for some of the processes already introduced in the examples of Section 2.2. Section 3.4.1 showed, via a lengthy proof, that the refraction strategy with rate $\delta < \infty$ is optimal when Π consists of absolutely continuous strategies whose densities are bounded by δ and X is the Cramér-Lundberg process with exponential jumps. The same result can be obtained for a Brownian motion with drift (see [2]). Since both processes are spectrally negative, and have completely monotone densities, theorem 21 yields identical results without requiring any computation. Theorem 21 is also extremely useful in calculating $v_{\pi_{b^*}}$ for both processes since both have manageable Laplace exponents.

Cramér-Lundberg Process

Recall that the Cramér-Lundberg Process with exponential jumps has Laplace exponent

$$\psi(z) = cz - \lambda \int_{-\infty}^{0} (1 - \mathrm{e}^{zx})\gamma \mathrm{e}^{\gamma x} dx \ .$$

One can integrate and combine terms on the right to obtain

$$\psi(z) = cz - \frac{\lambda z}{\gamma + z} \ .$$

The simple form of the Laplace exponent allows one to easily find roots to the equation $\psi(z) = q$, $q > 0$.

$$\begin{aligned} & q = cz - \frac{\lambda z}{\gamma + z} \\ \Leftrightarrow \quad & 0 = cz^2 + (\gamma c - \lambda - q)z - \gamma q \\ \Rightarrow \quad & z = \frac{(\lambda + q - \gamma c) \pm \sqrt{(\gamma c - \lambda - q)^2 + 4c\gamma q}}{2c} \end{aligned} \tag{3.58}$$

From (3.58) it follows that the the equation $\psi(z) = q$ has exactly two roots, both of which are real, one of which is positive, and one of which is negative. The positive root will be labelled $\Phi(q)$ and the negative root will be $-\zeta$. For the perturbed process $Y = \{Y_t = X_t - \delta t : t \geq 0\}$ the positive and negative roots will be $\varphi(q)$ and $-\widetilde{\zeta}$ respectively. Now, it is possible to find $W^q(x)$ explicitly by considering the inverse Laplace transform. This can be done using a partial fractions technique.

$$\frac{1}{\psi(z) - q} = \frac{\gamma + z}{c(z - \Phi(q))(z + \zeta)} = \frac{A}{(z - \Phi(q))} + \frac{B}{(z + \zeta)} \tag{3.59}$$

In (3.59), $A = (\Phi(q) + \gamma)/c(\zeta + \Phi(q))$ and $B = (\zeta - \gamma)/c(\zeta + \Phi(q))$ (for the perturbed process $\widetilde{A}$ and $\widetilde{B}$ will be used to denote the coefficients). The resulting expressions have known inverse Laplace transforms so that,

$$W^{(q)}(x) = \mathcal{L}^{-1}\left\{\frac{A}{(z - \Phi(q))}\right\}(x) + \mathcal{L}^{-1}\left\{\frac{B}{(z + \zeta)}\right\}(x) = A\mathrm{e}^{\Phi(q)x} + B\mathrm{e}^{-\zeta x} \ , \text{ and} \tag{3.60}$$

$$W^{(q)\prime}(x) = A\Phi(q)\mathrm{e}^{\Phi(q)x} - B\zeta\mathrm{e}^{-\zeta x} \ . \tag{3.61}$$

Now one can use (3.60), (3.61), and (3.46) to find an exact expression for $v_{\pi^{b*}}$. For presentation purposes it is easiest to evaluate (3.46) in pieces.

$$-\delta\int_0^{x-b^*}\mathbb{W}^{(q)}(y)dy = -\delta\left[\frac{\widetilde{A}\left(\mathrm{e}^{\varphi(q)(x-b^*)}-1\right)}{\varphi(q)}+\frac{\widetilde{B}\left(1-\mathrm{e}^{-\widetilde{\zeta}(x-b^*)}\right)}{\widetilde{\zeta}}\right] \tag{3.62}$$

$$\begin{aligned}\frac{\delta\int_{b^*}^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy}{h(b^*)} = \frac{\delta}{h(b^*)}&\left[\frac{\widetilde{A}A\Phi(q)\left(\mathrm{e}^{(\Phi(q)-\varphi(q))x}-\mathrm{e}^{(\Phi(q)-\varphi(q))b^*}\right)\mathrm{e}^{\varphi(q)x}}{(\Phi(q)-\varphi(q))}\right.\\
&+\frac{A\widetilde{B}\Phi(q)\left(\mathrm{e}^{(\Phi(q)+\widetilde{\zeta})x}-\mathrm{e}^{(\Phi(q)+\widehat{\zeta})b^*}\right)\mathrm{e}^{-\widetilde{\zeta}x}}{(\Phi(q)+\widetilde{\zeta})}\\
&+\frac{\widetilde{A}B\zeta\left(\mathrm{e}^{-(\varphi(q)+\zeta)x}-\mathrm{e}^{-(\varphi(q)+\zeta)b^*}\right)\mathrm{e}^{\varphi(q)x}}{(\varphi(q)+\zeta)}\\
&\left.+\frac{\widetilde{B}B\zeta\left(\mathrm{e}^{(\widetilde{\zeta}-\zeta)x}-\mathrm{e}^{(\widetilde{\zeta}-\zeta)b^*}\right)\mathrm{e}^{-\widetilde{\zeta}x}}{(\zeta-\widetilde{\zeta})}\right]\end{aligned} \tag{3.63}$$

Then $v_{\pi_{b^*}}(x)$ is the sum of (3.62), (3.63), and $W^{(q)}(x)/h(b^*)$ when $x > b^*$. Note, that $v_{\pi_{b^*}}(x) = W^{(q)}(x)/h(b^*)$ for $0 \le x \le b^*$ since $W^{(q)}(y) = 0$ for all $y < 0$. Although the resulting expression is not necessarily as elegant as (3.34) the use of theorem 21 clearly simplifies the process for finding $v_{\pi_{b^*}}$. Theorem 21 also gives clear direction as to when $b^* > 0$. Note that for this process $\mathrm{d} = -a - \int_{(-1,0)} x\nu(dx) = (c + \int_{(-1,0)} x\nu(dx)) - \int_{(-1,0)} x\nu(dx) = c$, so $b^* > 0$ precisely when $\varphi(q) < \delta(\nu(-\infty,0)+q)/c(c-q)$. Some algebra shows that,

$$\begin{aligned}
&\varphi(q) < \delta(\nu(-\infty,0)+q)/c(c-q)\\
\Leftrightarrow\ &\frac{(\lambda+q-\gamma(c-\delta))+\sqrt{(\gamma(c-\delta)-\lambda-q)^2+4(c-\delta)\gamma q}}{2(c-\delta)} < \frac{\delta(\lambda+q)}{c(c-\delta)}\\
\Leftrightarrow\ & c\sqrt{(\gamma(c-\delta)-\lambda-q)^2+4(c-\delta)\gamma q} < 2\delta(\lambda+q)-c(\lambda+q-\gamma(c-\delta))\\
\Leftrightarrow\ & c^2((\gamma(c-\delta)-\lambda-q)^2+4(c-\delta)\gamma q) < 4\delta^2(\lambda+q)^2\\
&\qquad -4c\delta(\lambda+q)(\lambda+q-\gamma(c-\delta))\\
&\qquad +c^2(\lambda+q-\gamma(c-\delta))^2\\
\Leftrightarrow\ & (c-\delta)\gamma q c^2 < \delta^2(\lambda+q)^2
\end{aligned}$$

$$- c\delta(\lambda+q)(\lambda+q-\gamma(c-\delta))$$

$$\Leftrightarrow \quad (c-\delta)\gamma q c^2 < \delta^2(\lambda+q)^2 - c\delta(\lambda+q)^2 + c\delta\gamma(\lambda+q)(c-\delta)$$

$$\Leftrightarrow \quad (c-\delta)\gamma q c^2 < \delta(\lambda+q)^2(\delta-c) + c\delta\gamma(\lambda+q)(c-\delta)$$

$$\Leftrightarrow \quad \gamma q c^2 < -\delta(\lambda+q)^2 + c\delta\gamma(\lambda+q)$$

$$\Leftrightarrow \quad \gamma q c^2 < \delta(\lambda+q)(c\gamma-\lambda-q)\ . \tag{3.64}$$

A comparison of the inequality (3.64) and the conditions following (3.45) show that they prescribe exactly the same behaviour for b^* in terms of the original parameters of the problem. When (3.64) holds, that is when $b^* > 0$, one can find the value of b^* by minimizing h. To do so, one can calculate

$$\begin{aligned} h(b) &= \varphi(q)\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)y} W^{(q)'}(y)dy \\ &= \varphi(q)\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)y}\left(A\Phi(q)\mathrm{e}^{\Phi(q)y} - B\zeta\mathrm{e}^{-\zeta y}\right)dy \\ &= \varphi(q)\left(\frac{A\Phi(q)\mathrm{e}^{\Phi(q)b}}{(\varphi(q)-\Phi(q))} - \frac{B\zeta\mathrm{e}^{-\zeta b}}{(\varphi(q)+\zeta)}\right), \end{aligned} \tag{3.65}$$

(3.66)

so that,

$$h'(b) = \varphi(q)\left(\frac{A\Phi(q)^2\mathrm{e}^{\Phi(q)b}}{(\varphi(q)-\Phi(q))} + \frac{B\zeta^2\mathrm{e}^{-\zeta b}}{(\varphi(q)+\zeta)}\right)\ . \tag{3.67}$$

Setting $h'(b) = 0$ and solving yields

$$b^* = \frac{\ln\left(\frac{\zeta^2(\gamma-\zeta)(\varphi(q)-\Phi(q))}{\Phi(q)^2(\gamma+\Phi(q))(\varphi(q)+\zeta)}\right)}{(\Phi(q)+\zeta)}\ . \tag{3.68}$$

Brownian Motion

In an earlier example it was established that the characteristic exponent of the scaled Brownian motion with drift has the form

$$\Psi(z) = -izc + \frac{1}{2}z^2\sigma^2.$$

This translates into the following Laplace exponent

$$\psi(z) = zc + \frac{1}{2}z^2\sigma^2. \tag{3.69}$$

Setting (3.69) equal to q and solving for z yields two real roots, one positive and one negative, that have the form

$$z = -\frac{c}{\sigma^2} \pm \sqrt{\frac{c^2 + 2q}{\sigma^2}}. \tag{3.70}$$

The positive and negative roots will be denoted by $\Phi(q)$ and $-\zeta$ respectively. For the perturbed process $Y = \{Y_t = X_t - \delta t : t \geq 0\}$ the equivalent expressions will be $\varphi(q)$ and $-\widetilde{\zeta}$. To find an explicit expression for $W^{(q)}$ one must consider the function $1/(\psi(z) - q)$ which one can decompose as follows:

$$\frac{1}{\psi(z) - q} = \frac{1}{1/2\sigma^2(z+\zeta)(z - \Phi(q))} = \frac{A}{z - \Phi(q)} + \frac{B}{z + \zeta}. \tag{3.71}$$

The constants A and B have the form $A = 2/(\sigma^2(\Phi(q)+\zeta))$ and $B = -A$. For the perturbed process the coefficients will be labelled $\widetilde{A}$ and $\widetilde{B}$. Applying the inverse Laplace transform to (3.71) yields

$$W^{(q)}(x) = \mathcal{L}^{-1}\left\{\frac{A}{(z - \Phi(q))}\right\}(x) + \mathcal{L}^{-1}\left\{\frac{B}{(z+\zeta)}\right\}(x) = A\mathrm{e}^{\Phi(q)x} + B\mathrm{e}^{-\zeta x}\text{ , so that} \tag{3.72}$$

$$W^{(q)\prime}(x) = A\Phi(q)\mathrm{e}^{\Phi(q)x} - B\zeta\mathrm{e}^{-\zeta x}\ . \tag{3.73}$$

Upon comparing (3.72) and (3.73) with (3.60) and (3.61) one observes that the form of the expressions is identical. Of course, it is understood that the constants in the equations represent different values. Thus $v_{\pi_{b^*}}$ will have exactly the same form as for the Cramér-Lundberg process. Note that as in the Cramér-Lunberg case the term d is just equal to the drift of the process c. Thus when $\varphi(q) < \delta(\nu(-\infty, 0) + q)/c(c - q)$ one can calculate b^* in the same manner as in the Cramér-Lundberg scenario. This yields

$$b^* = \frac{\ln\left(\frac{\zeta^2(\varphi(q)-\Phi(q))}{\Phi(q)^2(\varphi(q)+\zeta)}\right)}{(\Phi(q) + \zeta)} \; . \tag{3.74}$$

Remark No effort has been made to reconcile these new formulas to existing formulas for $v_{\pi_{b^*}}$ or b^* found in [2] for the Brownian Motion case, or in [13] for the Cramér-Lundberg case. Numerical evaluations support the fact that they are equivalent.

4 Numerical Analysis

4.1 The Distribution of $\sigma^{\pi_{b^*}}$ for the Cramér-Lundberg Process

A relevant and important question that appears often in papers dealing with optimal dividend strategies is how to characterize $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ the random variable that represents the firm's time to ruin under strategy π. Ideally, one is interested in finding the distribution of σ^π which is often a difficult or impossible task. In lieu of finding the distribution other useful quantities and identities can usually be determined. For example, the following quantity is often of interest,

$$R(x) = \mathbb{P}_x\left(\sigma^\pi < \infty\right).$$

The function $R(x)$ is just the probability that ruin occurs in finite time, a concept which is otherwise known as the *infinite horizon ruin probability.* The related function $S(x) = 1 - R(x)$ is the *infinite horizon survival probability*, or the probability that ruin does not occur in finite time. Often, exact expressions can be determined for R and S. For example, in section VII.1 of [1] Asmussen finds a closed form expression for $R(x)$ when the underlying risk process is a refracted Cramér-Lundberg process with exponential jumps. In other words, it is possible to calculate the infinite horizon ruin/survival probability when the optimal refraction strategy is applied to the Cramér-Lundberg process.

Another identity that is often calculated in association with this problem is

$$L(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\sigma^\pi}\right].$$

The financial relevance of L is that it represents the present value of one dollar paid at the time of ruin. In similar fashion (and using the same notation) to the calculation of the value function in section (3.4.1) Gerber and Shiu [13] show that $L(x)$ has the following form when X is the Cramér-Lundberg process controlled by the optimal refraction strategy with barrier b^*:

$$L(x) = \begin{cases} \frac{1}{\gamma}\frac{(\gamma+\Phi)(\zeta_1-\zeta_2)\mathrm{e}^{\Phi x-\zeta_1 b^*}+(\gamma-\zeta_1)(\Phi+\zeta_2)\mathrm{e}^{-\zeta_1 x+\Phi x}}{(\Phi+\zeta_2)\mathrm{e}^{\zeta_1 b^*}+(\zeta_1-\zeta_2)\mathrm{e}^{-\Phi b^*}}, & 0 \le x \le b^* \\ L(b^*)\mathrm{e}^{-\zeta_2(x-b^*)}, & x \ge b^*. \end{cases}$$

It has been demonstrated several times that results for the Cramér-Lundberg process can be extended to spectrally negative processes in general. This is almost the case for $L(x)$. One can calculate an explicit expression for $L(x)$ in terms of scale functions when X is spectrally negative process and π is the reflection (not refraction) strategy. To give the result, it is necessary to introduce the notation $\overline{X}_t = \sup_{0\leq s\leq t} X_t$. Also, let $Y = x + X$ so that Y has the property $Y_0 = x$. Note that the reflection strategy with barrier b^* applied to the process Y yields,

$$U_t^{\pi} = Y_t - (\overline{Y}_t - b^* \vee 0)$$

Therefore one can write σ^{π} for the process Y as

$$\begin{aligned}
\sigma^{\pi} &= \inf\{t > 0 : Y_t - (\overline{Y}_t - b^* \vee 0) < 0\} \\
&= \inf\{t > 0 : Y_t - (\overline{Y}_t \vee b^*) + b^* < 0\} \\
&= \inf\{t > 0 : ((x + \overline{X}_t) \vee b^*) - (x + X_t) > b^*\} \\
&= \inf\{t > 0 : (\overline{X}_t \vee (b^* - x)) - X_t) > b^*\} = \overline{\sigma}_{b^*}^{b^*-x}
\end{aligned}$$

The value $\overline{\sigma}_{b^*}^{b^*-x}$ is known as the exit time for the process reflected in its supremum. Chapter 8, pg. 216 of [20] then gives the following identity:

$$L(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\sigma^{\pi}}\right] = \mathbb{E}_x\left[\mathrm{e}^{-q\overline{\sigma}_{b^*}^{b^*-x}}\right] = Z^{(q)}(x) - W^{(q)}(x)\frac{W^{(q)}(b^*)}{W^{(q)\prime}(b^*)}, \quad x \in [0, b^*],$$

where $Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy$, and $W^{(q)}$ is the scale function of the process X.

One can also get a sense of σ^{π} for the refraction strategy by simulating the process $U^{\pi_{b^*}}$ and developing empirical estimates of its distribution. The Cramér-Lundberg process with exponential jumps lends itself well to such simulation. Upon deciding on a relevant drift rate c, one can choose the parameter λ for the Poisson process (determining the frequency of the jumps), and the parameter γ for the random variables ξ_n (determining the size of the jumps). With this, one can calculate b^* via (3.74) and simulate two sets of exponential random variables, one set with parameter λ which models the times between events of the Poisson process, and one with parameter γ which models the jump sizes. The exponential random variables can be generated using the formula

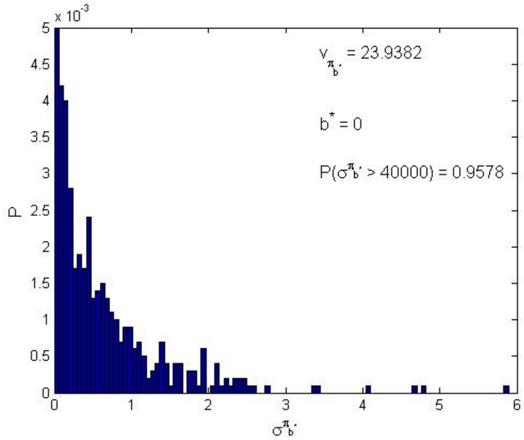
$$Z = \frac{-\log(U(0,1))}{k},$$

where $U(0,1)$ is a uniform random variable simulated using a random number generator on $[0,1]$ and k is either λ or γ. The π_{b^*} controlled process $U^{\pi_{b^*}}$ is then constructed by applying the modified drift $c-\delta$ once b^* has been attained. After a suitably large number of jumps of the Poisson process the simulation is stopped. If ruin has occurred the time is recorded. This procedure can be repeated and the data aggregated to give empirical estimates of the distribution of $\sigma^{\pi_{b^*}}$. The graphs in figure 5 were created using this procedure with 10,000 simulations and 10,000 Poisson jumps. The programs necessary to carry out the simulation were written in Matlab and can be found in Appendix A and B.

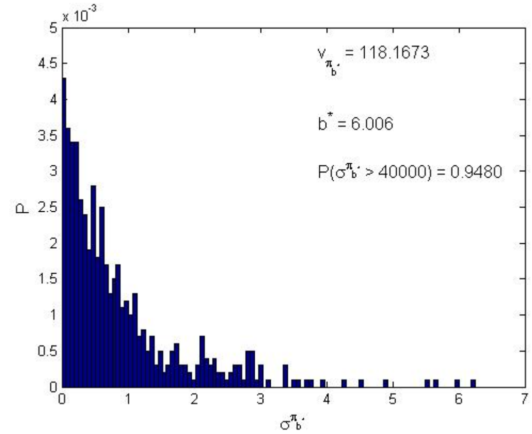
The graphs in figure 5 support the intuition that aggressive strategies, i.e. those that pay large amounts of incoming capital above b^*, are much more likely to cause ruin in finite time. Notice that for the last simulation $\delta \approx c$, so that the optimal refraction strategy is practically a reflection strategy. Of course, the infinite horizon ruin probability can be calculated exactly via the formula for $R(x)$ found in [1]. An interesting question is: Which strategy offers the highest reward when adjusted for its risk of ruin? For example, comparing figures 5a and 5b one notices a marked difference in the value of $v_{\pi_{b^*}}$, but almost no difference in the empirical distribution for $\sigma^{\pi_{b^*}}$. In deciding between optimal strategies it seems that the strategy with $\delta = 2.5$ is the clear winner over the strategy with $\delta = 0.5$. The answer to the question is not as obvious when comparing the strategies depicted in figures 5f and 5g. The value of $v_{\pi_{b^*}}$ does not increase as dramatically with the increase in δ, and the empirical distribution has a significantly different character. Optimizing a dividend strategy with respect to overall dividend payment and also risk is a potential avenue for further research.

4.2 The Value Function for Meromorphic Processes

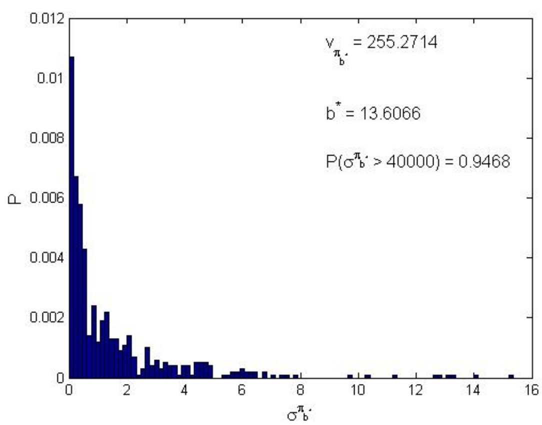
This section will combine the theory of meromorphic processes introduced in section 2.4 and the results of theorem 21 to approximate value functions for the refraction strategy for meromorphic processes with Laplace exponents given by (2.34), (2.35), and (2.39). The approach to finding



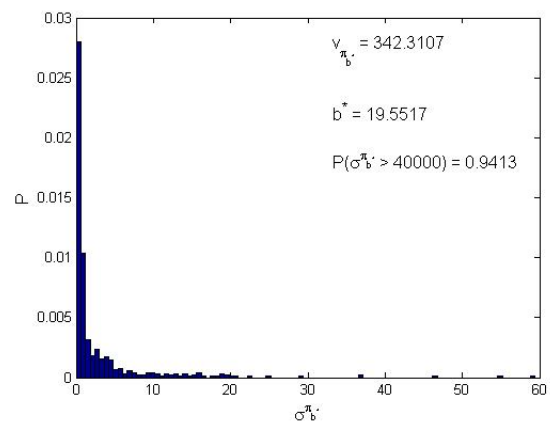
(a) $\delta = 0.5$



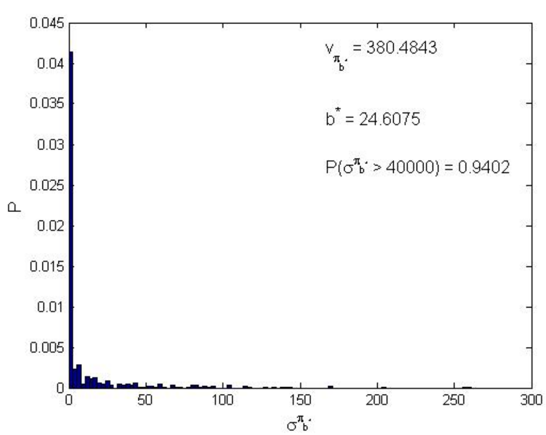
(b) $\delta = 2.5$



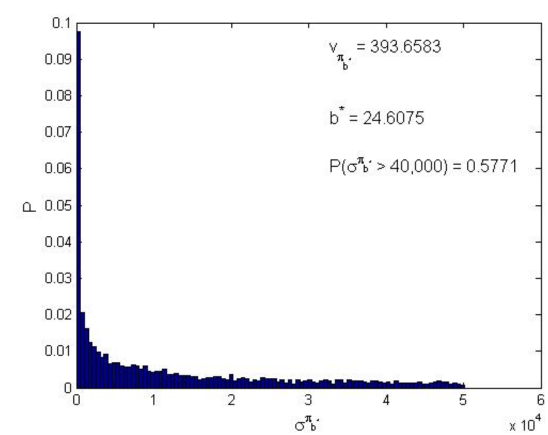
(c) $\delta = 5.5$



(d) $\delta = 7.5$

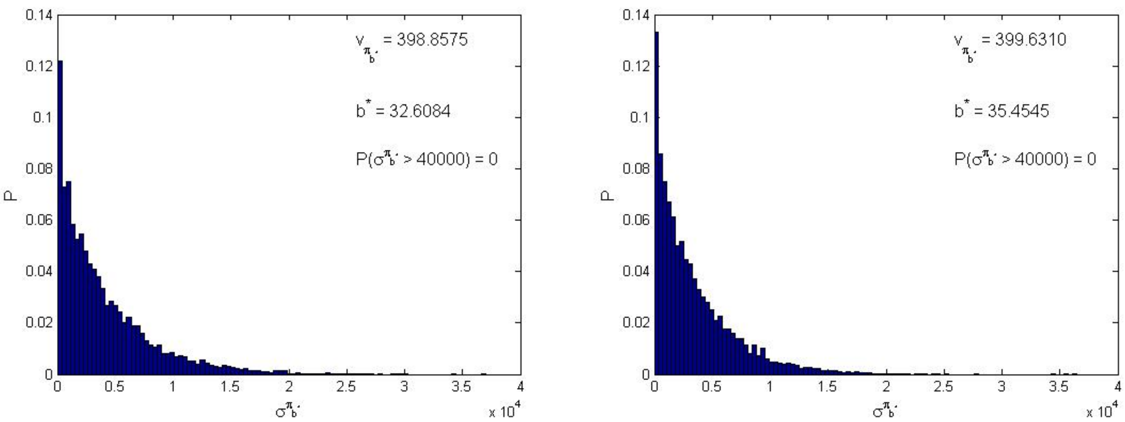


(e) $\delta = 8.5$



(f) $\delta = 9$

Figure 5



(g) $\delta = 9.5$ (h) $\delta = 9.999$

Figure 5: Comparison of optimal refraction strategies for the Cramér-Lundberg process with $x = 5$, $c = 10$, $\lambda = 0.2$, $\gamma = 0.2$, and $q = 0.02$.

approximations for $v_{\pi_{b^*}}$ will be very similar to that used to find exact values for the Cramér-Lundberg process or the Brownian motion. The need for approximation arises because the scale functions for the three chosen types of processes are expressed as infinite sums, and because it is not possible to find the exact roots of the equation $\psi(z) = q$. Since the numerical techniques were all performed using computer programs, references will be given to the relevant code whenever possible.

Remark 1 In the following sections the series expressions for $W^{(q)}$ and $W^{(q)'}$ (it will be shown that $W^{(q)'}$ can also be expressed as a series) will be substituted into the formulas for h, and $v_{\pi_{b^*}}$. At this point, the integrals in both h and $v_{\pi_{b^*}}$ can be evaluated so that h and $v_{\pi_{b^*}}$ can be written as the sum of one or more infinite sums. In all cases the summands of the series will include an expression that behaves like

$$\mathrm{e}^{-\beta n x} \text{ or } \mathrm{e}^{-\beta n^2 x}. \tag{4.1}$$

When the exponents in (4.1) exceed 50, additional summands are essentially 0 and contribute little more to the overall sum. Thus, all series expressions in $W^{(q)}$, $W^{(q)'}$, h or $v_{\pi_{b^*}}$ are truncated after $n = 50/(\beta x)$ terms as a means of approximation. For very small values of x this makes computations more challenging since the number of terms increases very quickly.

Remark 2 All of the series expressions in the following calculations derive from the scale function,

$$W^{(q)}(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{\mathrm{e}^{-\zeta_n x}}{\psi'(-\zeta_n)}, \tag{4.2}$$

where $\Phi(q)$ is the positive solution and $\{-\zeta_n\}_{n\geq 1}$ the negative solutions of $\psi(z) = q$.

Recall from proposition 10 that (4.2) was derived by applying inverse Laplace transforms to both sides of the expression

$$\frac{1}{\psi(z) - q} = \frac{1}{\psi'(\Phi(q))(z - \Phi(q))} + \sum_{n=1}^{\infty} \frac{1}{\psi'(-\zeta_n)(z + \zeta_n)}. \tag{4.3}$$

Also recall from the proof of theorem 8 that $\psi'(-\zeta_n) \leq 0$. Setting $z = 0$ in (4.3) then shows that

$$\sum_{n=1}^{\infty} \frac{1}{\psi'(-\zeta_n)(\zeta_n)}$$

is a convergent series. This, in turn, implies that there exists N such that

$$\frac{-1}{\psi'(-\zeta_n)} < \zeta_n$$

for all $n \geq N$. Using the same technique as in proposition 5 one can show that series of the form

$$\sum_{n\geq 1} (-\zeta_n)^m \frac{\mathrm{e}^{-\zeta_n x}}{\psi'(-\zeta_n)} \tag{4.4}$$

converge uniformly on intervals (ε, ∞) for all $m \in \mathbb{N}$ and $\varepsilon > 0$. This finding provides a sufficient condition to differentiate the series in(4.2) term-by-term to obtain

$$W^{(q)'}(x) = \frac{\Phi(q)\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} - \sum_{n\geq 1} \frac{\zeta_n \mathrm{e}^{-\zeta_n x}}{\psi'(-\zeta_n)}. \tag{4.5}$$

It also justifies interchanging integration and summation whenever one is asked to integrate $W^{(q)}(x)$

or $W^{(q)'}(x)$ over intervals where $x > 0$. Finally, since either

$$(-\zeta_n)^m \frac{\mathrm{e}^{-\zeta_n x}}{\psi'(-\zeta_n)} \geq 0 \text{ for all } n \geq 1 \quad \text{or} \quad (-\zeta_n)^m \frac{\mathrm{e}^{-\zeta_n x}}{\psi'(-\zeta_n)} \leq 0 \text{ for all } n \geq 1$$

it is clear that (4.4) converges absolutely. As a result the product of two series of the form (4.4) can be expressed as an infinite double sum where the order of summation is irrelevant. Since the partial sums of the resulting double sum will form a monotone sequence, integration of such a double sum can be performed by interchanging integration and summation. Justification for this procedure is given by the well known monotone convergence theorem.

4.2.1 Approximating $W^{(q)}$ and $W^{(q)'}$

In order to generate approximations for (4.2) and (4.5) it will be necessary to solve numerically for $\Phi(q)$ and $\{-\zeta_n\}_{n\geq 1}$. Recall that the processes in question have Laplace exponents of the form

- θ-process with parameter $\lambda = 3/2$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z - c\sqrt{\alpha + z/\beta}\coth\left(\pi\sqrt{\alpha + z/\beta}\right) + c\sqrt{\alpha}\coth\left(\pi\sqrt{\alpha}\right),$$

- θ-process with parameter $\lambda = 5/2$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + c\left(\alpha + z/\beta\right)^{\frac{3}{2}}\coth\left(\pi\sqrt{\alpha + z/\beta}\right) - c\alpha^{\frac{3}{2}}\coth\left(\pi\sqrt{\alpha}\right),$$

- β-process with parameter $\lambda \in (0, 3)\backslash\{1, 2\}$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + c\mathrm{B}\left(1 + \alpha + z/\beta, 1 - \lambda\right) - c\mathrm{B}\left(1 + \alpha, 1 - \lambda\right).$$

These have derivatives:

- θ-process with parameter $\lambda = 3/2$

$$\psi'(z) = \sigma^2 z + \mu - \frac{c}{2\beta}\left(\frac{\coth\left(\pi\sqrt{\alpha + z/\beta}\right)}{\sqrt{\alpha + z/\beta}} - \pi\mathrm{csch}^2\left(\pi\sqrt{\alpha + z/\beta}\right)\right),$$

- θ-process with parameter $\lambda = 5/2$

$$\psi'(z) = \sigma^2 z + \mu + \frac{c(\alpha + z/\beta)}{2\beta} \left(\frac{3\coth\left(\pi\sqrt{\alpha + z/\beta}\right)}{\sqrt{\alpha + z/\beta}} - \pi\text{csch}^2\left(\pi\sqrt{\alpha + z/\beta}\right) \right),$$

- β-process with parameter $\lambda \in (1,2) \cup (2,3)$

$$\psi'(z) = \sigma^2 z + \mu + \frac{c}{\beta} \left(\mathrm{B}\left(1 + \alpha + z/\beta, 1 - \lambda\right) \left(\mathrm{D}\left(1 + \alpha + z/\beta\right) - \mathrm{D}\left(2 + \alpha - \lambda + z/\beta\right)\right)\right).$$

In the last expression, B represents the beta function and D represents the digamma function ($D(x) = \Gamma'(x)/\Gamma(x)$).

Finding the negative roots $\{-\zeta_n\}_{n\geq 1}$ is greatly simplified by the result of theorem 8 which guarantees that ζ_{n+1} is located in the interval

$$(\rho_n, \rho_{n+1}) = \left(\beta(\alpha + n^2), \beta(\alpha + (n+1)^2)\right), \quad n \geq 1, \text{ for theta processes,}$$
$$(\rho_n, \rho_{n+1}) = \left(\beta(\alpha + n), \beta(\alpha + (n+1))\right), \quad n \geq 1, \text{ for beta processes, and}$$
$$\zeta_1 \in (0, \rho_1).$$

As a result one can quickly approximate the values of the negative roots of $\psi(z) = q$ using the bisection method started just inside the endpoints of these intervals. Finding the positive root $\Phi(q)$ requires slightly more effort, but can ultimately be accomplished via bisection following a quick search. See Appendices D.5, D.2, D.4 for the computer program used to generate the poles, find the negative roots, and find the positive root respectively. Once the roots have been established one can easily form the truncated series expressions for $W^{(q)}$ and $W^{(q)\prime}$ using the derivatives of the Laplace exponents calculated above.

4.2.2 Approximating b^* and $h(b^*)$

Consider first the case where $b^* > 0$, that is, either $\sigma = 0$, or $\sigma > 0$ and $\varphi(q) < 2\delta/\sigma^2$. Theorem 21 provides a guarantee that b^* is the unique point at which h attains its minimum. Thus, proceeding as in the Cramér-Lundberg or Brownian motion case, one can differentiate h with respect to b, set the resulting expression equal to 0 and solve for b^*. For beta and theta processes, the result will be an approximation since it is necessary to express h in terms of the truncated series for $W^{(q)'}$, which in turn is expressed in terms of the approximated roots of $\psi(z) = q$.

As indicated, one can differentiate $h(b)$ with respect to b and simplify.

$$
\begin{aligned}
h'(b) &= \varphi(q)^2 e^{\varphi(q)b} \int_b^\infty e^{-\varphi(q)y} W^{(q)'}(y)dy - \varphi(q) W^{(q)'}(b) \\
&= \varphi(q)^2 e^{\varphi(q)b} \int_b^\infty \frac{\Phi(q) e^{(\Phi(q)-\varphi(q))y}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{-\zeta_n e^{-(\varphi(q)+\zeta_n)y}}{\psi'(-\zeta_n)} dy \\
&\quad - \varphi(q) \left(\frac{\Phi(q) e^{\Phi(q)b}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{-\zeta_n e^{-\zeta_n b}}{\psi'(-\zeta_n)} \right) \\
&= \varphi(q)^2 e^{\varphi(q)b} \left(\frac{\Phi(q) e^{(\Phi(q)-\varphi(q))y}}{(\Phi(q)-\varphi(q))\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{\zeta_n e^{-(\varphi(q)+\zeta_n)y}}{(\varphi(q)+\zeta_n)\psi'(-\zeta_n)} \Bigg|_b^\infty \right) \\
&\quad - \varphi(q) \left(\frac{\Phi(q) e^{\Phi(q)b}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{-\zeta_n e^{-\zeta_n b}}{\psi'(-\zeta_n)} \right) \\
&= -\varphi(q)^2 e^{\varphi(q)b} \left(\frac{\Phi(q) e^{(\Phi(q)-\varphi(q))b}}{(\Phi(q)-\varphi(q))\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{\zeta_n e^{-(\varphi(q)+\zeta_n)b}}{(\varphi(q)+\zeta_n)\psi'(-\zeta_n)} \right) \\
&\quad - \varphi(q) \left(\frac{\Phi(q) e^{\Phi(q)b}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{-\zeta_n e^{-\zeta_n b}}{\psi'(-\zeta_n)} \right) \\
&= -\varphi(q)^2 \frac{\Phi(q) e^{\Phi(q)b}}{(\Phi(q)-\varphi(q))\psi'(\Phi(q))} - \varphi(q) \frac{\Phi(q) e^{\Phi(q)b}}{\psi'(\Phi(q))} \\
&\quad - \varphi(q)^2 \sum_{n\geq 1} \frac{\zeta_n e^{-\zeta_n b}}{(\varphi(q)+\zeta_n)\psi'(-\zeta_n)} - \varphi(q) \sum_{n\geq 1} \frac{-\zeta_n e^{-\zeta_n b}}{\psi'(-\zeta_n)} \\
&= -\varphi(q) \frac{\Phi(q) e^{\Phi(q)b}}{\psi'(\Phi(q))} \left(\frac{\varphi(q)}{\Phi(q)-\varphi(q)} + 1 \right) \\
&\quad + \varphi(q) \sum_{n\geq 1} \left(\frac{\zeta_n e^{-\zeta_n b}}{\psi'(-\zeta_n)} \left(\frac{-\varphi(q)}{\varphi(q)+\zeta_n} + 1 \right) \right)
\end{aligned}
$$

$$= \sum_{n\geq 1}\left(\frac{\varphi(q)\zeta_n^2 \mathrm{e}^{-\zeta_n b}}{\psi'(-\zeta_n)(\varphi(q)+\zeta_n)}\right) - \frac{\varphi(q)\Phi(q)^2\mathrm{e}^{\Phi(q)b}}{\psi'(\Phi(q))(\Phi(q)-\varphi(q))}$$

Note that in the fourth equality, the fact $\Phi(q)-\varphi(q) < 0$ is used to show that the integral converges. To approximate b^* one can search for an interval in $(0,\infty)$ on which the approximation of h' changes sign and then proceed with bisection. See appendix D.8 for the program used to complete this calculation.

Once one has a value for b^* one can evaluate $h(b^*)$. The derivation of $h(b^*)$ is very similar to that of $h'(b)$. The result is given by

$$h(b^*) \quad = \quad \varphi(q)\left(\frac{\Phi(q)\mathrm{e}^{b^*\Phi(q)}}{(\varphi(q)-\Phi(q))\,\psi'(\Phi(q))} + \sum_{n\geq 1}\frac{-\zeta_n \mathrm{e}^{-b^*\zeta_n}}{(\zeta_n+\varphi(q))\,\psi'(-\zeta_n)}\right)$$

Now, consider the case where $b^* = 0$, that is, suppose $\sigma = 0$ and $\varphi(q) \geq 2\delta/\sigma^2$. Recall from (3.51) that when $b^* = 0$,

$$v_{\pi_0}(x) = -\delta\left(\int_0^x \mathbb{W}^{(q)}(y)dy - \frac{1}{\varphi(q)}\mathbb{W}^{(q)}(x)\right). \tag{4.6}$$

As a result, it is not necessary to calculate $h(0)$ in order to evaluate v_{π_0}.

4.2.3 Approximating $v_{\pi_{b^*}}$

The goal is to approximate

$$v_{\pi_{b^*}}(x) = -\delta\int_0^{x-b^*}\mathbb{W}^{(q)}(y)dy + \frac{W^{(q)}(y)}{h(b^*)} + \frac{\delta\int_{b^*}^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy}{h(b^*)},\ x \geq 0 \tag{4.7}$$

using the approximations already established. When $x \leq b^*$ the value function $v_{\pi_{b^*}}(x)$ is simply equal to $W^{(q)}(x)/h(b^*)$ which can be easily be calculated using the previous results. All further analysis will proceed under the assumption that $x > b^*$.

The following calculations will involve the potential measure $\hat{u}^{(q)}(x)dx$ of the dual process first introduced in section (2.3). Recall from (2.18) and (2.19) that the density of the potential measure of the dual process is a bounded function that has the form

$$\hat{u}^{(q)}(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} - W^{(q)}(x) \ , \tag{4.8}$$

and satisfies

$$F^{(q)}(z) = \int_0^\infty \mathrm{e}^{-zx}\hat{u}^{(q)}(x)dx = \frac{1}{\psi'(\Phi(q))(z-\Phi(q))} - \frac{1}{\psi(z)-q} \ , \tag{4.9}$$

for $\Phi(q) > 0$ and $z \geq 0$.

Before proceeding it is necessary to establish some additional notation. Recall that $\mathbb{W}^{(q)}$ is the scale function for the linearly perturbed process and that $\varphi(q)$ is the largest positive root of the equation $\psi(z) - \delta z = q$. In the following calculations a "$\sim$" superscript will denote an expression that derives from the perturbed process. For example, $\{-\widetilde{\zeta}_n\}_{n\geq 1}$ are the negative roots of $\widetilde{\psi}(z) = \psi(z) - \delta z = q$. The two exceptions to this will be $\varphi(q)$ which has already been defined, and $\hat{\mathbf{u}}^{(q)}$ which will denote the density of the dual of the potential measure of the perturbed process.

What remains is to find manageable expressions for the first and third terms of the right hand side of (4.7). In order to maintain oversight over the calculations this will be done individually for each of the expressions. First, consider the case $b^* > 0$, then

$$\begin{aligned}
-\delta \int_0^{x-b^*} \mathbb{W}^{(q)}(y)dy &= -\delta \int_0^{x-b^*} \frac{\mathrm{e}^{\varphi(q)y}}{\widetilde{\psi}'(\varphi(q))} - \hat{\mathbf{u}}^{(q)}(y)dy \\
&= -\delta \left(\int_0^{x-b^*} \frac{\mathrm{e}^{\varphi(q)y}}{\widetilde{\psi}'(\varphi(q))}dy - \int_0^\infty \hat{\mathbf{u}}^{(q)}(y)dy + \int_{x-b^*}^\infty \hat{\mathbf{u}}^{(q)}(y)dy \right) \\
&= -\delta \left(\int_0^{x-b^*} \frac{\mathrm{e}^{\varphi(q)y}}{\widetilde{\psi}'(\varphi(q))}dy - \widetilde{F}^{(q)}(0) + \int_{x-b^*}^\infty \sum_{n\geq 1} \frac{\mathrm{e}^{-\widetilde{\zeta}_n y}}{\widetilde{\psi}'(-\widetilde{\zeta}_n)}dy \right) \\
&= -\delta \left(\frac{\mathrm{e}^{\varphi(q)(x-b^*)}}{\varphi(q)\widetilde{\psi}'(\varphi(q))} - \sum_{n\geq 1} \frac{\mathrm{e}^{-\widetilde{\zeta}_n(x-b^*)}}{\widetilde{\zeta}_n\widetilde{\psi}'(-\widetilde{\zeta}_n)} - \frac{1}{q} \right) ,
\end{aligned} \tag{4.10}$$

where identity (4.9) is used in the third equality of (4.10). Next, continuing under the assumption that $b^* > 0$,

$$\frac{\delta \int_{b^*}^{x} \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)dy}{h(b^*)} = \frac{\delta}{h(b^*)}\left(\int_{b^*}^{x}\left(\frac{\mathrm{e}^{\varphi(q)(x-y)}}{\widetilde{\psi}'(\varphi(q))} - \hat{\mathbf{u}}^{(q)}(x-y)\right)\left(\frac{\Phi(q)\mathrm{e}^{\Phi(q)y}}{\psi'(\Phi(q))} - \hat{u}^{(q)\prime}(y)\right)dy\right)$$
$$= \frac{\delta}{h(b^*)}\left(\int_{b^*}^{x}\frac{\mathrm{e}^{\varphi(q)(x-y)}}{\widetilde{\psi}'(\varphi(q))}\frac{\Phi(q)\mathrm{e}^{\Phi(q)y}}{\psi'(\Phi(q))}dy - \int_{b^*}^{x}\frac{\Phi(q)\mathrm{e}^{\Phi(q)y}}{\psi'(\Phi(q))}\hat{\mathbf{u}}^{(q)}(x-y)dy\right.$$
$$\left. - \int_{b^*}^{x}\frac{\mathrm{e}^{\varphi(q)(x-y)}}{\widetilde{\psi}'(\varphi(q))}\hat{u}^{(q)\prime}(y)dy + \int_{b^*}^{x}\hat{\mathbf{u}}^{(q)}(x-y)\hat{u}^{(q)\prime}(y)dy\right) \tag{4.11}$$

For presentation purposes, consider each of the integrals in (4.11) separately. The first and third integrals are straightforward, and one can calculate the second integral using a similar approach as in (4.10):

$$\int_{b^*}^{x}\frac{\Phi(q)\mathrm{e}^{\Phi(q)y}}{\psi'(\Phi(q))}\hat{\mathbf{u}}^{(q)}(x-y)dy = \frac{\Phi(q)\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))}\int_{0}^{x-b^*}\mathrm{e}^{-\Phi(q)w}\hat{\mathbf{u}}^{(q)}(w)dw$$
$$= \frac{\Phi(q)\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))}\left(\int_{0}^{\infty}\mathrm{e}^{-\Phi(q)w}\hat{\mathbf{u}}^{(q)}(w)dw - \int_{x-b^*}^{\infty}\mathrm{e}^{-\Phi(q)w}\hat{\mathbf{u}}^{(q)}(w)dw\right)$$
$$= \frac{\Phi(q)\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))}\left(\widetilde{F}^{(q)}(\Phi(q)) + \int_{x-b^*}^{\infty}\sum_{n\geq 1}\frac{\mathrm{e}^{-(\Phi(q)+\zeta_n)w}}{\widetilde{\psi}'(-\widetilde{\zeta}_n)}dw\right)$$
$$= \frac{\mathrm{e}^{\Phi(q)x}\Phi(q)}{\psi'(\Phi(q))}\left(\widetilde{F}^{(q)}(\Phi(q)) + \sum_{n\geq 1}\frac{\mathrm{e}^{-(x-b^*)(\Phi(q)+\widetilde{\zeta}_n)}}{(\Phi(q)+\widetilde{\zeta}_n)\widetilde{\psi}'(-\widetilde{\zeta}_n)}\right). \tag{4.12}$$

Equation (4.12) makes use of identity (4.9) in the third equality. Finally, the fourth integral in (4.11) can be calculated as follows:

$$\int_{b^*}\hat{\mathbf{u}}^{(q)}(x-y)\hat{u}^{(q)\prime}(y)dy = -\sum_{n\geq 1}\sum_{m\geq 1}\int_{b^*}^{x}\frac{\zeta_n\mathrm{e}^{-(x-y)\widetilde{\zeta}_m}\mathrm{e}^{-\zeta_n y}}{\widetilde{\psi}'(-\widetilde{\zeta}_m)\psi'(-\zeta_n)}dy$$
$$= -\sum_{n\geq 1}\sum_{m\geq 1}\frac{\zeta_n(\mathrm{e}^{-\zeta_n x} - \mathrm{e}^{-\widetilde{\zeta}_m x}\mathrm{e}^{b^*(\widetilde{\zeta}_m-\zeta_n)})}{(\widetilde{\zeta}_m-\zeta_n)\widetilde{\psi}'(-\widetilde{\zeta}_m)\psi'(-\zeta_n)}.$$

The complete expression for $v_{\pi_{b^*}}$ is then given by 4.13. The formula is broken into smaller expressions each of which is evaluated using a separate program. Each of the pieces is labelled with the title of the appendix section that lists the appropriate program's code.

$$
\begin{aligned}
v_{\pi_{b^*}}(x) = {}& -\delta \overbrace{\left(\frac{\mathrm{e}^{\varphi(q)(x-b^*)}}{\varphi(q)\widetilde{\psi}'(\varphi(q))} - \sum_{n\geq 1} \frac{\mathrm{e}^{-\widetilde{\zeta}_n(x-b^*)}}{\widetilde{\zeta}_n \widetilde{\psi}'(-\widetilde{\zeta}_n)} - \frac{1}{q} \right)}^{D.11} + \frac{\overbrace{W^{(q)}(x)}^{D.12}}{h(b^*)} \\
& + \frac{\delta}{h(b^*)} \left(\frac{\mathrm{e}^{\varphi(q)x}}{\widetilde{\psi}'(\varphi(q))} \left(\overbrace{\frac{\Phi(q)(\mathrm{e}^{x(\Phi(q)-\varphi(q))} - \mathrm{e}^{b^*(\Phi(q)-\varphi(q))})}{(\Phi(q)-\varphi(q))\psi'(\Phi(q))}}^{D.13} \right.\right. \\
& \left. + \overbrace{\sum_{n\geq 1} \frac{\zeta_n \left(\mathrm{e}^{-x(\varphi(q)+\zeta_n)} - e^{-b^*(\varphi(q)+\zeta_n)}\right)}{(\varphi(q)+\zeta_n)\psi'(-\zeta_n)}}^{D.14} \right) \\
& - \frac{\mathrm{e}^{\Phi(q)x}\Phi(q)}{\psi'(\Phi(q))} \overbrace{\left(\widetilde{F}^{(q)}(\Phi(q)) + \sum_{n\geq 1} \frac{\mathrm{e}^{-(x-b^*)(\Phi(q)+\widetilde{\zeta}_n)}}{(\Phi(q)+\widetilde{\zeta}_n)\widetilde{\psi}'(-\widetilde{\zeta}_n)} \right)}^{D.15} \\
& \left. - \overbrace{\sum_{n\geq 1}\sum_{m\geq 1} \frac{\zeta_n(\mathrm{e}^{-\zeta_n x} - \mathrm{e}^{-\widetilde{\zeta}_m x}\mathrm{e}^{b^*(\widetilde{\zeta}_m-\zeta_n)})}{(\widetilde{\zeta}_m-\zeta_n)\widetilde{\psi}'(-\widetilde{\zeta}_m)\psi'(-\zeta_n)}}^{D.16} \right)
\end{aligned}
\tag{4.13}
$$

When $b^* = 0$ the calculation is greatly simplified. In this case, one can use expression (4.6) to obtain

$$
v_{\pi_0}(x) = -\delta \left(\overbrace{\left(\frac{\mathrm{e}^{\varphi(q)x}}{\varphi(q)\widetilde{\psi}'(\varphi(q))} - \sum_{n\geq 1} \frac{\mathrm{e}^{-\widetilde{\zeta}_n x}}{\widetilde{\zeta}_n \widetilde{\psi}'(-\widetilde{\zeta}_n)} - \frac{1}{q} \right)}^{D.11} - \overbrace{\frac{1}{\varphi(q)}\mathbb{W}^{(q)}(x)}^{D.12} \right). \tag{4.14}
$$

See Appendices C and D for a full listing of the programs used to calculate $v_{\pi_{b^*}}$.

4.2.4 Discussion of the Results

A valid question is: Does the technique described in the previous section produce an accurate approximation of $v_{\pi_{b^*}}$? Although no rigorous error analysis is provided here, there are two simple tests one can conduct to gain an understanding as to whether the approximation behaves like the actual value function. The first method of verification comes from condition (3.50) which states that $v'_{\pi_{b^*}}(x) \geq 1$ if $x \leq b^*$, and that $v'_{\pi_{b^*}}(x) \leq 1$ if $x \geq b^*$. The second method derives from the knowledge that $\lim_{x\to\infty} v_{\pi_{b^*}}(x) = \delta/q$, and that $v_{\pi_{b^*}}(x) \leq \delta/q$ for all x. Figure 6 depicts the approximated value function for a theta process with parameters $\sigma = 0.25$, $\mu = 24$, $\delta = 4$, $c = 4$, $\alpha = 0.5$, $\beta = 1$, $q = 0.02$, and $\lambda = 5/2$. As desired, f, the line through point $(b^*, v_{\pi_{b^*}}(b^*))$ with slope 1, is tangent to the approximation of $v_{\pi_{b^*}}$ at the point b^*. Further, for all $x \leq b^*$, the slopes of the tangent lines at $(x, v_{\pi_{b^*}}(x))$ are less than 1, and for all $x \geq b^*$ these slopes are greater than 1. Additionally, as x increases $v_{\pi_{b^*}}(x)$ increases, gradually approaching, but never exceeding, the asymptote δ/q.

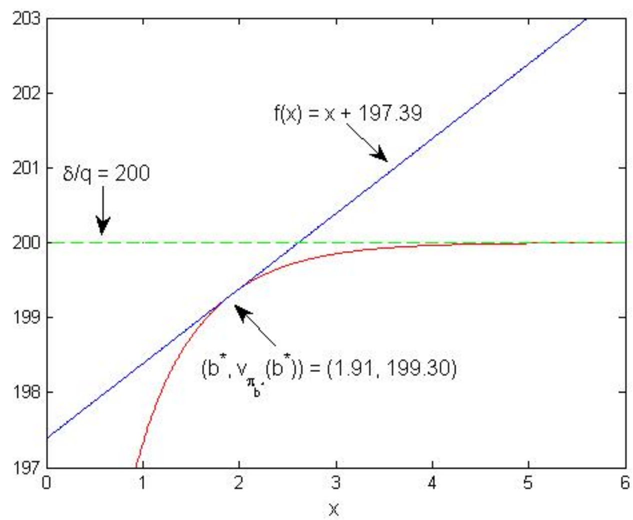
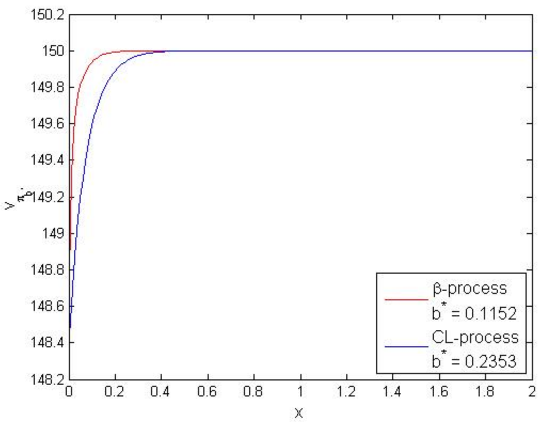
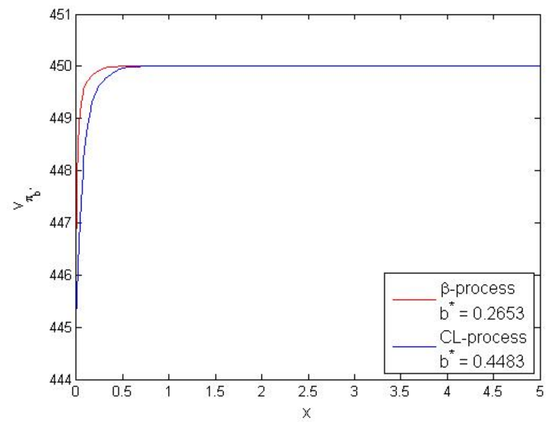


Figure 6: Tests of the validity of the approximation for a theta process with parameters $\sigma = 0.25$, $\mu = 24$, $\delta = 4$, $c = 4$, $\alpha = 0.5$, $\beta = 1$, $q = 0.02$, and $\lambda = 5/2$.

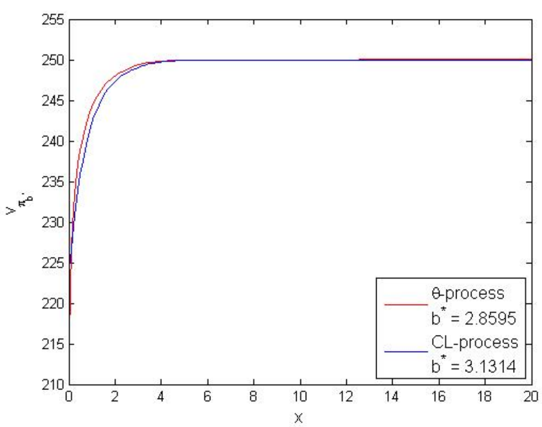
As mentioned in the introduction, researchers in actuarial science are divided on the issue of whether it is necessary to approach actuarial problems in the fully general setting of spectrally negative Lévy processes. Some argue that simple models like the Cramér-Lundberg process give computationally satisfactory results when substituted for the more complicated processes discussed in this section. Figures 7a – 7b compare the approximated $v_{\pi_{b^*}}$ for a beta process with parameters $\sigma = 0.1$, $\mu = 10$, $c = 0.21$, $\alpha = 13.87$, $\beta = 1$, $q = 0.02$, and $\lambda = 3/2$ with the value function of a Cramér-Lundberg process with exponential jumps with parameters $c = 10$, $\lambda = 3/2$, and $\gamma = 14.87$. Figures 7c – 7d make the same comparison for a theta process with parameters $\sigma = 0$, $\mu = 10$, $c = 2.26$, $\alpha = 0.53$,



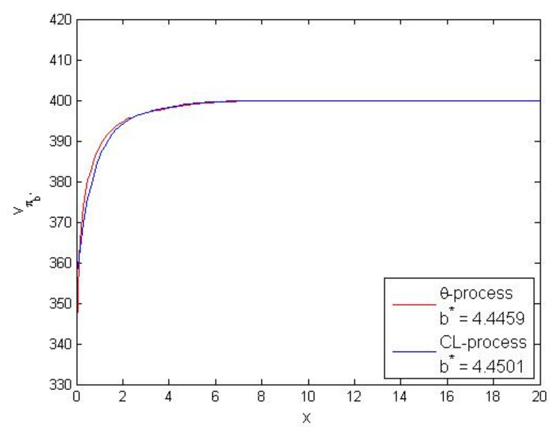
(a) $\delta = 3$



(b) $\delta = 9$



(c) $\delta = 5$



(d) $\delta = 8$

Figure 7: Comparison of Cramér-Lundberg process with beta and theta processes having the same expectation and variance. The top graphs compare the value function of a beta process with parameters $\sigma = 0.1$, $\mu = 10$, $c = 0.21$, $\alpha = 13.87$, $\beta = 1$, $q = 0.02$, and $\lambda = 3/2$ to the value function of a Cramér-Lundberg process with exponential jumps with parameters $c = 10$, $\lambda = 3/2$, and $\gamma = 14.87$. The bottom graphs make the same comparison for a theta process with parameters $\sigma = 0$, $\mu = 10$, $c = 2.26$, $\alpha = 0.53$, $\beta = 1$, $q = 0.02$, and $\lambda = 3/2$ and a Cramér-Lundberg process with parameters $c = 10$, $\lambda = 3/2$, and $\gamma = 1.53$. Note that the scale is different for each graph.

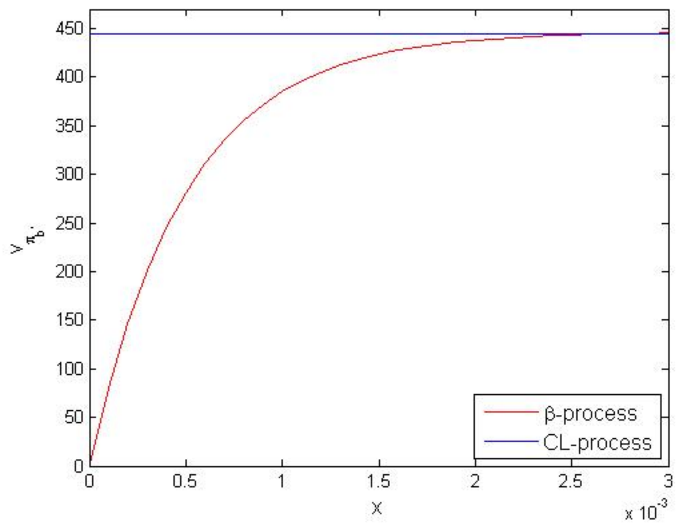


Figure 8: Demonstration of a poor approximation of the value function for values of x near 0 for the beta process and associated Cramér-Lundberg process from figure 7.

$\beta = 1$, $q = 0.02$, and $\lambda = 3/2$ and a Cramér-Lundberg process with parameters $c = 10$, $\lambda = 3/2$, and $\gamma = 1.53$. In both cases, the parameters have been chosen so that the expectation and variance of the beta or theta process are the same as the expectation and variance of the Cramér-Lundberg process. Additionally, the condition $\beta(1+\alpha) = \gamma$ has been imposed so that the Lévy measures of the processes exhibit the same behaviour at negative infinity. In both scenarios the approximation given by the Cramér-Lundberg process is quite good, especially for larger values of x. Even in the first case where the underlying beta process has paths of infinite activity and infinite variation the value function of the Cramér-Lundberg process provides a reasonable facsimile for almost all x.

If x is small, however, the behaviour of the value function for infinite variation and finite variation processes differs significantly. Recall that when $b^* > 0$, $v_{\pi_{b^*}} = W^{(q)}(x)/h(b^*)$ for all $x \leq b^*$. Further, recall that $W^{(q)}(0) = 0$ for processes with paths of infinite variation, but that $W^{(q)}(0) = 1/\mathrm{d}$ for processes with paths of finite variation. As a result, the value function of the Cramér-Lundberg process does not provide a good approximation for the beta process when x is near 0. Figure 8 depicts the value functions for the same beta process and Cramér-Lundberg process as figure 7b for values of x near the origin.

5 Conclusion and Suggested Future Work

This thesis has provided a review of the current state of research on the optimal dividend problem. Additionally, it has demonstrated the practical application of theorem 21 through the use of two families of M-processes and standard numerical techniques. In an ancillary discussion it has considered the distribution for $\sigma^{\pi_{b^*}}$ for a Cramér-Lundberg process with exponentially distributed jumps. This discussion indicated that an interesting avenue of research may be to optimize dividends not only with respect to total value, but also with respect to the risk of ruin. The optimization problem in this case may be to maximize expressions of the form

$$v_\pi(x) = \mathbb{E}_x\left[\int_{[0,\sigma^\pi]} \mathrm{e}^{-qt} dL_t^\pi - f(\sigma^\pi)\right],$$

where f is a decreasing penalty function. Determining the appropriate function may be the most challenging part of this exercise.

Additionally, it may be interesting to test the model through an empirical study. That is, for a sample of insurance companies one could fit appropriate gross wealth processes based on market data and compare $v_{\pi_{b^*}}$ to the companies' share prices. Ideally, one would hope to find that the model accurately predicts current prices.

Finally, one can modify the optimal dividend problem and consider instead the problem which has a finite expiry time T (see for example, the proof of lemma 11). A possible approach to solving this problem may be to use the technique of randomization, first introduced by Carr [7], to value American put options. Carr approximates the value of an American put with expiry T by considering the same problem with a random exponentially distributed expiry τ that has expectation T. This problem has a closed form solution. To improve the accuracy of the estimate, he considers the multi-step problem with expiry $T' = \sum_{i=1}^n \tau_i$, where τ_n is also an exponential random variable, now with expectation T/n, so that $\mathbb{E}[T'] = T$ and $\mathrm{Var}(T') = T^2/n$. As $n \to \infty$, the variance of the gamma random variable T' approaches 0 and its density converges to a Dirac delta function centred at T while the approximation for the value function converges to the true value function of the American put with expiry T. The unknown exercise boundary of the American put option is somehow analogous to the unknown barrier of an optimal dividend strategy for the problem with infinite horizon. Thus, it seems possible that the randomization technique may yield favourable results when applied to the finite expiry optimal dividend problem.

Appendices

A Code Hierarchy for Ruin Time Simulations

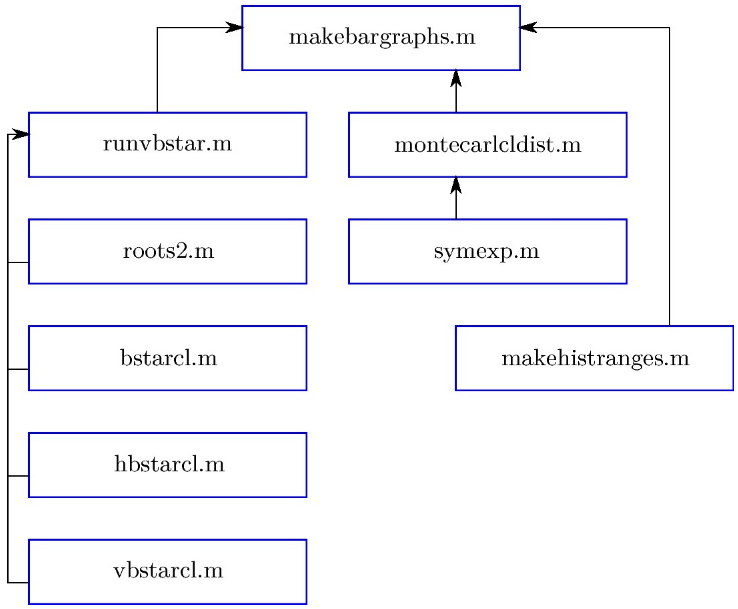


Figure 9: An arrow from a program indicates this program is called by the program to which the arrow points.

B Matlab Code for Ruin Time Simulations

B.1 makebargraphs.m

```
function [u,v,yu,TMAX] = makebargraphs(mu,delta,lambda,GAM,q,x,n,m)

% n is number of poisson jumps, m is number of bars in bar graph

% run vbstar to find bstar

[vbs,bstr] = runvbstar(mu,delta,lambda,GAM,q,x(1));

% run the monte carlo simulations. s does not include total times of paths not ruined,
% while t does

[s,t] = montecarlcldist(mu,delta,x,lambda,GAM,bstr,n);
f = (s == 0);
M = max(s);
N = min(t(f));

% make the ranges for the bar chart

[w,ex,y] = makehistranges(s,m,M)

% make barchart
y1 = y(1:(length(y)-1));
bar(y1,w,'histc');

% output

u = vbs;     % vbstar output
v = bstr;    % bstar output
yu = ex;     % percentage greater than TMAX
TMAX = N;
```

B.2 runvbstar.m

```
function [u,v] = runvbstar(mu,delta,lambda,GAM,q,x)

% runvbstar for the cramer lundberg process

% make the unperturbed roots
[lphiq,nrt] = roots2(lambda,q,GAM,mu,0);

% make the unperturbed coefficients A and B
A = (lphiq + GAM)/(mu*(lphiq - nrt));
B = (-nrt - GAM)/(mu*(lphiq - nrt));

% make the perturbed roots
[sphiq,nrth] = roots2(lambda,q,GAM,mu,delta);

% make the perturbed coefficients Ah and Bh
Ah = (sphiq + GAM)./((mu-delta)*(sphiq - nrth));
Bh = (-nrth - GAM)./((mu-delta)*(sphiq - nrth));

% make bstar
bstr = bstarcl(lphiq,nrt,sphiq,GAM);

% make hbstar
hbstr = hbstarcl(lphiq,nrt,sphiq,A,B,bstr);

% execute vbstarcl
u = vbstarcl(lphiq,nrt,sphiq,nrth,A,B,Ah,Bh,bstr,hbstr,delta,x);
v = bstr;
```

B.3 roots2.m

```
function [s,t] = roots2(lambda,q,GAM,mu,delta)
```

```
% finds the positive and negative roots for the perturbed and unpertrurbed process

s = ((lambda + q − GAM*(mu − delta)) + sqrt((GAM*(mu − delta) − lambda −q).^2 ...
    + 4*(mu−delta)*GAM*q))./(2*(mu − delta));
t = ((lambda + q − GAM*(mu − delta)) − sqrt((GAM*(mu − delta) − lambda −q).^2 ...
    +4*(mu−delta)*GAM*q))./(2*(mu − delta));
```

B.4 bstarcl.m

```
function u = bstarcl(lphiq,nrt,sphiq,GAM)

% calculates bstar exactly for the cramer lundberg process

u = max(0,log(((nrt^2)*(sphiq − lphiq)*(GAM + nrt))...
    ./((lphiq^2)*(sphiq − nrt)*(GAM + lphiq)))/(lphiq − nrt));
```

B.5 hbstarcl.m

```
function u = hbstarcl(lphiq,nrt,sphiq,A,B,bstar)

u = sphiq*(((A*lphiq*exp(lphiq*bstar))/(sphiq − lphiq))...
     − ((B*(−nrt)*exp(nrt*bstar))/(sphiq − nrt)));
```

B.6 vbstarcl.m

```
function u = vbstrcl(lphiq,nrt,sphiq,nrth,A,B,Ah,Bh,bstr,hbstr,delta,x)

% divide x into pieces bigger and smaller than bstar
y = x <= bstr;
```

```
z = x > bstr;
y1 = x(y);
y2 = x(z);
t = zeros(size(x));

% when x <= bstar
S1 = (A*exp(lphiq*y1) + B*exp(nrt*y1))/hbstr;

% make first section

S2 = -delta*(((Ah/sphiq)*(exp(sphiq*(y2 - bstr))-1))...
    + ((Bh/(-nrth))*(1-exp(nrth*(y2 - bstr)))));

% make second section

S3 = (delta./hbstr).*...
    (((Ah*A*lphiq.*(exp((lphiq - sphiq)*y2)...
    - exp((lphiq - sphiq).*bstr)).*exp(sphiq*y2))./(lphiq - sphiq))...
    + ((A*Bh*lphiq.*(exp((lphiq - nrth)*y2)...
    - exp((lphiq - nrth).*bstr)).*exp(nrth*y2))./(lphiq - nrth))...
    + ((Ah*B*(-nrt).*(exp(-(sphiq - nrt)*y2)...
    - exp(-(sphiq - nrt).*bstr)).*exp(sphiq*y2))./(sphiq - nrt))...
    + ((Bh*B*(-nrt).*(exp((nrt - nrth)*y2)...
    - exp((nrt - nrth).*bstr)).*exp(nrth*y2))./(nrth - nrt)));

% make last section

S4 = (A*exp(lphiq*y2) + B*exp(nrt*y2))/hbstr;

% put it together

t(y) = S1;
t(z) = S2 + S3 + S4;
u = t;
```

B.7 montecarlcldist.m

```
function [u,v] = montecarlcldist(mu,delta,x,lambda,GAM,bstr,n)

% A vector x has many entries all with the same value symbolizing starting
% capital. Then length(x) simulations take place that model the net wealth
% process under the optimal barrier strategy (refraction up to relection).
% This process is repeated for n time steps. The resulting vector can be
% used to make a histogram that will approximate the distribution of the
% time to ruin.

m = length(x);
K = zeros(1,m);
H = zeros(1,m);
tot = x;
for i = 1:n;
    % identify those that are already out of range
    f = (tot < 0);

    % simulate exponential random variables for the time step t
    % and the jump j
    t = symexp(m,lambda);
    j = symexp(m,GAM);

    % identify those that are greater than bstar and were originally
    % in range (b)
    b = ((tot >= bstr) & ~f);

    % identify those that become greater than bstar in the
    % current period and were not already bigger than bstar or were out
    % of range (c)
    c = (((tot + mu*t) >= bstr) & ~b & ~f);

    % identify those that are less than bstar including the current
    % period and were originally in range (d)
```

```
    d = (((tot + mu*t) < bstr) & ~f);

    % increment the bs by the reduced drift and subtract the jump
    tot(b) = tot(b) + ((mu - delta).*t(b)) - j(b);

    % find the time (tmp) when the cs cross bstar and increment the
    % cs by mu*tmp and the reduced drift for the remaining time then
    % subtract the jump
    tmp = (bstr - tot(c))/mu;
    tot(c) = tot(c) + mu*tmp + ((mu - delta).*(t(c) - tmp)) - j(c);

    % increment the ds by the full drift and subtract the jump
    tot(d) = tot(d) + mu*t(d) - j(d);

    % identify those that are still in range
    g = (tot > 0);

    % add the time step to their cumulative time
    K(g) = K(g) + t(g);

    % identify those that left the range in this time step
    h = ((tot < 0) & ~f);

    % add the time step to their cumulative time
    K(h) = K(h) + t(h);

    % record the time of ruin for those that have left
    H(h) = K(h);
end
u = H;
v = K;
```

B.8 symexp.m

```
function u = symexp(k,a)
```

```
u = −log(rand(1,k))/a;
```

B.9 makehistranges.m

```
function [u,v,w] = makehistranges(x,m,MAX1)

% normalizes the vector to make probability plot

n = length(x);
q = linspace(0,MAX1,m);
k = zeros(size(1,m−1));
for i = 1:m−1
    h = q(i) < x & x <= q(i+1);
    l = sum(h);
    k(i) = l;
end
r = (x == 0);
u = k/n;
v = sum(r)/n;
w = q;
```

C Code Hierarchy for Value Function Calculations

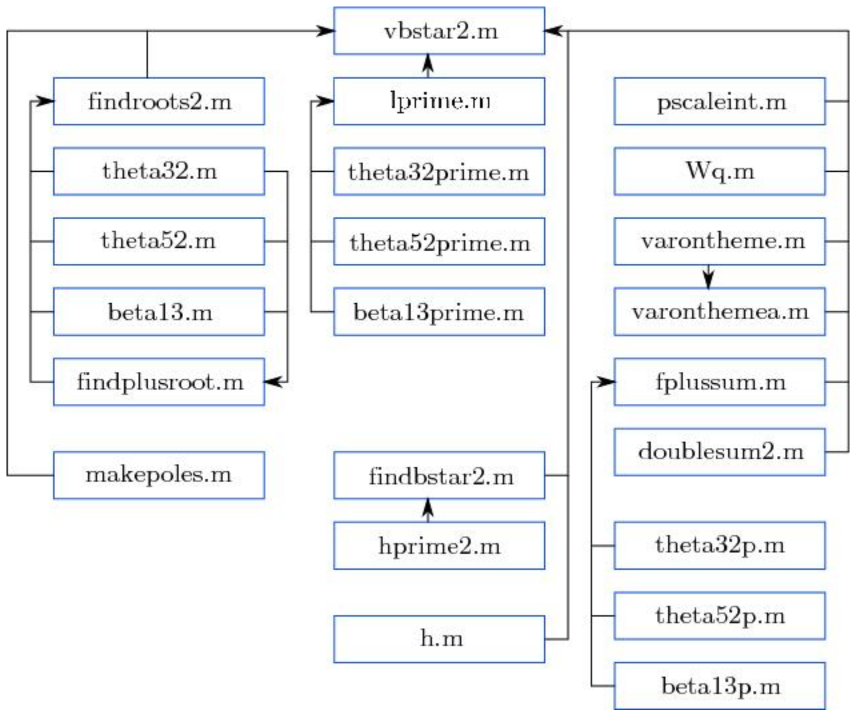


Figure 10: An arrow from a program indicates this program is called by the program to which the arrow points.

D Matlab Code for Value Function Calculations

D.1 vbstar2.m

```
function v = vbstar2(sigma,mu,delta,c,alpha,beta,q,torb,lambda,k,x)

% evaluates everything to find vbstar

if sigma < 0
    error('sigma must be nonnegative!');
elseif c <= 0
    error('c must be positive!');
elseif alpha <= 0
    error('alpha must be positive!');
elseif beta <= 0
    error('beta must be positive!');
elseif q < 0
    error('q must be positive!');
elseif torb ~= 0 && torb ~= 1 && torb ~= 2
    error('torb must be 0,1,or 2!');

% 0 corresponds to a theta process with parameter lambda = 3/2
% 1 corresponds to a theta process with parameter lambda = 5/2
% 2 corresponds to a beta process with parameter lamba in (1,2) or (2,3)

elseif lambda <= 1 || lambda == 2 || lambda >= 3
    error('lambda is not in range!');
else

% first make poles corresponding to the correct process

    if torb == 0 || torb == 1
        [s,t] = makepoles(alpha,beta,0,k);
    else
        [s,t] = makepoles(alpha,beta,1,k);
```

```
    end

% find lphiq the positive root of the unperturbed process, and nrts
% the negative roots of the unperturbed process

    [lphiq, nrts] = findroots2(sigma,mu,0,c,alpha,beta,q,torb,lambda,k,s,t);

% find latlhphiq the derivative of laplace exponent evaluated at the positive
% root lphiq, and latnrts the derivative of the laplace exponent evaluated
% at the negative roots nrts

    [latlphiq, latnrts] = lprime(sigma,mu,0,c,alpha,beta,lambda,torb,lphiq,nrts);

% do the same for the perturbed process (h stands for hat)
% note that the only difference is that the formula below now includes delta

    [sphiq, nrtsh] = findroots2(sigma,mu,delta,c,alpha,beta,q,torb,lambda,k,s,t);
    [latsphiq, latnrtsh] = lprime(sigma,mu,delta,c,alpha,beta,lambda,torb,sphiq,nrtsh);

% find bstar

    bstar = findbstar2(sigma,delta,nrts,lphiq,sphiq,latlphiq,latnrts)

% find h(bstar)

    if bstar > 0
        hbstar = h(lphiq,bstar,sphiq,nrts,latlphiq,latnrts);
    end

% calculate vbstar

    R = ones(size(x));
    a = (x > bstar);
    b = (x <= bstar);
    if bstar > 0
        if max(a) == 1
        R(a) = -delta*pscaleint(sphiq,latsphiq,nrtsh,latnrtsh,bstar,x(a),q) ...
```

```
        + Wq(lphiq,latlphiq,nrts,latnrts,x(a))/hbstar ...
        + (delta/hbstar)*( ...
            (exp(sphiq*x(a))/latsphiq).*( ...
                lphiq*varontheme((lphiq - sphiq),bstar*(lphiq - sphiq),...
                (lphiq - sphiq),latlphiq,x(a))...
                + varonthemea(-(sphiq - nrts),-bstar*(sphiq - nrts),...
                (sphiq - nrts),latnrts,nrts,x(a))...
            )...
            -(exp(lphiq*x(a))*lphiq/latlphiq)...
            .*fplussum(sigma,mu,delta,c,alpha,beta,q,torb,lambda,...
            lphiq,sphiq,latsphiq,nrtsh,latnrtsh,bstar,x(a))...
            - doublesum2(nrts,nrtsh,latnrts,latnrtsh,bstar,x(a)) ...
        );
        end
        if max(b) == 1
        R(b) = Wq(lphiq,latlphiq,nrts,latnrts,x(b))/hbstar;
        end
    else
        if max(a) == 1
        R(a) = -delta*(pscaleint(sphiq,latsphiq,nrtsh,latnrtsh,bstar,x(a),q)...
        - ((1/sphiq)*(Wq(sphiq,latsphiq,nrtsh,latnrtsh,x(a),q))));
        end
        if max(b) == 1
        R(b) = Wq(lphiq,latlphiq,nrts,latnrts,x(b))/hbstar;
        end
    end

    v = R;

end
```

D.2 findroots2.m

Note, some of the following program has been omitted. The section that is shown is for theta processes with $\lambda = 3/2$. The omitted sections perform the same calculations for the other two types of processes.

```
function [u,v] = findroots2(sigma,mu,delta,c,alpha,beta,q,torb,lambda,k,s,t)

% Returns a vector with the negative roots of the equation of
% the Laplace exponent and a scalar for the one positive root
% when the Laplace exponent is equal to some q > 0

% run bisection using poles from correct process

    vs = zeros(size(s));
    vt = zeros(size(t));
    half = (t+s)/2;

    if torb == 0
        while abs(max(s-t)) > 1.0e-10
            vs = theta32(sigma,s,mu,delta,c,alpha,beta,q);
            vt = theta32(sigma,t,mu,delta,c,alpha,beta,q);
            if any(vs.*vt >= 0)
                error('there is not necessarily a root between two of the points!');
            else
                r = vs.*theta32(sigma,half,mu,delta,c,alpha,beta,q);
                n = (r > 0);
                m = (r < 0);
                p = (r == 0);
                s(n) = half(n);
                t(m) = half(m);
                s(p) = half(p);
                t(p) = half(p);
            end
            half = (t+s)/2;
        end
```

```
    v = half;
    u = findplusroot(sigma,mu,delta,c,alpha,beta,q,torb,lambda);
end
```

D.3 theta32.m, theta52.m, and beta13.m

This program calculates $\psi(z) - q$ for theta processes with $\lambda = 3/2$. Programs theta52.m, and beta13.m perform the same calculation for the other two types of processes. These are not shown here as the code is nearly identical.

```
function v = theta32(sigma,z,mu,delta,c,alpha,beta,q)

% Calculates psi(z) - q for the theta process with
% lambda = 3/2. For the unperturbed process set delta = 0.

r = zeros(size(z));
m = ((alpha + z/beta) > 0);
n = ((alpha + z/beta) < 0);
p = ((alpha + z/beta) == 0);
r(m) = (1/2)*(sigma^2)*(z(m).^2) + (mu-delta)*z(m) - c*((alpha + (z(m)/beta)).^(1/2))...
    .*coth(pi*((alpha + (z(m)/beta)).^(1/2)))...
    + c*(alpha^(1/2))*coth(pi*(alpha^(1/2))) - q;
r(n) = (1/2)*(sigma^2)*(z(n).^2) + (mu-delta)*z(n) + c*((-(alpha + z(n)/beta)).^(1/2))...
    .*cot(-pi*((-(alpha + z(n)/beta)).^(1/2)))...
    + c*(alpha^(1/2))*coth(pi*(alpha^(1/2))) - q;
r(p) = (1/2)*(sigma^2)*(z(p).^2) + (mu-delta)*z(p)...
    + c*(((alpha + z(p)/beta)-1.0e-10).^(1/2))...
    .*coth(-pi*(((alpha + z(p)/beta)-1.0e-10).^(1/2)))...
    + c*(alpha^(1/2))*coth(pi*(alpha^(1/2))) - q;
v = real(r);
```

D.4 findplusroot.m

Note, some of the following program has been omitted. The section that is shown is for theta processes with $\lambda = 3/2$. The omitted sections perform the same calculations for the other two types of processes.

```
function v = findplusroot(sigma,mu,delta,c,alpha,beta,q,torb,lambda)
```

```
% finds the positive root of the Laplace exponent

s = 0:10:10000;
t = 10:10:10010;

% select the correct process

if torb == 0

% find an interval on which the function changes sign

    while all(theta32(sigma,s,mu,delta,c,alpha,beta,q)...
        .*theta32(sigma,t,mu,delta,c,alpha,beta,q) > 0)
        t = t(1001):10:(t(1001)+s(1001));
        s = s(1001):10:(s(1001)+s(1001));
    end

% run bisection on the appropriate interval
% return error in the unlikely case that the root is one of the interval endpoints,
%or there is more than one interval on which the function changes sign

    r = theta32(sigma,s,mu,delta,c,alpha,beta,q).*theta32(sigma,t,mu,delta,c,alpha,beta,q);
    n = (r < 0 & isfinite(r));
    if any(r == 0)
        error('congratulations, somehow one of your roots is a positive integer!');
    elseif sum(n) > 1
        error('you have more than one positive root!');
    else
        s = s(n);
        t = t(n);
        half = (s+t)/2;
        while abs(max(s-t)) > 1.0e-10
            temp = theta32(sigma,s,mu,delta,c,alpha,beta,q)...
            .*theta32(sigma,half,mu,delta,c,alpha,beta,q);
            if temp > 0
                s = half;
```

```
            elseif temp < 0
                t = half;
            else
                s = half;
                t = half;
            end
            half = (s+t)/2;
        end
    end
```

```
    end
```

D.5 makepoles.m

```
function [v1,v2] = makepoles(a,b,c,k)

% Makes two vectors with poles of the Laplace exponent
% v1 and v2 are off-set by one position and contain k
% elements each.

v1 = zeros(1,k);
v2 = zeros(1,k);
t1 = 1.0e-10;
m = 1:k;

if c ~= 0 && c~= 1
    error('c must equal 0 or 1!');
elseif c == 0
    v2 = b*(a + (m.^2));
    v1(2:k) = v2(1:k-1);
    v1 = -v1 - t1;
    v2 = -v2 + t1;
else
```

```
    v2 = b*(a + m);
    v1(2:k) = v2(1:k-1);
    v1 = -v1 - t1;
    v2 = -v2 + t1;
end
```

D.6 lprime.m

```
function [u,v] = lprime(sigma,mu,delta,c,alpha,beta,lambda,torb,prt,nrts)

% Evaluates the unperturbed laplace exponent of the appropriate process
% at the positive and negative roots. Returns a scalar value for the positive
% root and a vector value for the negative roots.

if torb == 0;
u = theta32prime(sigma,prt,mu,delta,c,alpha,beta);
v = theta32prime(sigma,nrts,mu,delta,c,alpha,beta);
elseif torb == 1;
u = theta52prime(sigma,prt,mu,delta,c,alpha,beta);
v = theta52prime(sigma,nrts,mu,delta,c,alpha,beta);
else
u = beta13prime(sigma,prt,mu,delta,c,alpha,beta,lambda);
v = beta13prime(sigma,nrts,mu,delta,c,alpha,beta,lambda);
end
```

D.7 theta32prime.m, theta52prime.m, beta13prime.m

This program calculates $\psi'(z)$ for theta processes with $\lambda = 3/2$. Programs theta52prime.m, and beta13prime.m perform the same calculation for the other two types of processes. These are not shown here as the code is nearly identical.

```
function v = theta32prime(sigma,z,mu,delta,c,alpha,beta)
```

```
% Calculates the derivative of psi for the theta process with lambda = 3/2
% at specific values

r = ones(size(z));
if any((alpha + z/beta) == 0)
    error('function not defined at 0');
end
m = ((alpha + z/beta) > 0);
n = ((alpha + z/beta) < 0);
r(m) = (sigma^2)*z(m) + (mu-delta) - (c/(2*beta))...
    *(coth(pi*((alpha + z(m)/beta).^(1/2)))./((alpha + z(m)/beta).^(1/2))...
    - pi*((csch(pi*((alpha + z(m)/beta).^(1/2)))).^2));
r(n) = (sigma^2)*z(n) + (mu-delta) - (c/(2*beta))...
    *(cot(-pi*((-(alpha + z(n)/beta)).^(1/2)))./((-(alpha + z(n)/beta)).^(1/2))...
    + pi*((csc(-pi*((-(alpha + z(n)/beta)).^(1/2)))).^2));
v = real(r);
```

D.8 findbstar2.m

```
function v = findbstar2(sigma,delta,nrts,lphiq,sphiq,latlphiq,latnrts)

% finds bstar for any process

if (sigma > 0) && (sphiq >= 2*delta/(sigma^2))
    v = 0;
else
    u = hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,-1);
    i = 9;
    while u*hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,i) > 0
        i = i+10;
    end
    j = u*hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,i);
    if j == 0;
        error('great you found a root!')
    else
```

```
        s = i-10;
        t = i;
        half = (s+t)/2;
        while abs(max(s-t)) > 1.0e-10
            temp = hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,s)...
                .*hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,half);
            if temp > 0
                s = half;
            elseif temp < 0
                t = half;
            else
                s = half;
                t = half;
            end
            half = (s+t)/2;
        end
    end
    if half < 0
        v = 0;
    else
        v=half;
    end
end
```

D.9 hprime2.m

```
function v = hprime2(nrts,lphiq,sphiq,latlphiq,latnrts,b)

% Makes hprime for the appropriate process

r = ((nrts.^2).*exp(nrts*b))./(latnrts.*(sphiq - nrts));
s = ((lphiq^2)*exp(lphiq*b))/(latlphiq*(lphiq - sphiq));
v = sphiq*(sum(r)-s);
```

D.10 h.m

```
function v = h(lphiq,bstar,sphiq,nrts,latlphiq,latnrts)

% evaluates h(b*)

n = length(nrts)+1;
A = zeros(n,length(bstar));
A(1,:) = (lphiq*exp(bstar*lphiq))/((sphiq - lphiq)*latlphiq);
for i = 2:n
    A(i,:) = (nrts(i-1)*exp(bstar*nrts(i-1)))/((-nrts(i-1) + sphiq)*(latnrts(i-1)));
end
v = sphiq*sum(A);
```

D.11 pscaleint.m

```
function v = pscaleint(sphiq,latsphiq,nrtsh,latnrtsh,bstar,x,q)

% makes the pscaleint piece of vbstar

n = length(nrtsh)+1;
A = zeros(n,length(x));
A(1,:) = exp(sphiq*(x - bstar))/(sphiq*latsphiq);
for i = 2:n
    A(i,:) = exp(nrtsh(i-1)*(x - bstar))/(nrtsh(i-1)*latnrtsh(i-1));
end
v = sum(A) - (1/q);
```

D.12 Wq.m

```
function v = Wq(lphiq,latlphiq,nrts,latnrts,x)
```

```
% makes the scale function Wq

n = length(nrts)+1;
A = zeros(n,length(x));
A(1,:) = exp(lphiq*x)/latlphiq;
for i = 2:n
    A(i,:) = exp(nrts(i-1)*x)/latnrts(i-1);
end
v = sum(A);
```

D.13 varontheme.m

```
function v = varontheme(A,B,C,D,x)

% evaluates the common expression (e^xA - e^B)/CD in vbstar

v = (exp(x*A)-exp(B))/(C*D);
```

D.14 varonthemea.m

```
function v = varonthemea(A,B,C,D,nrts,x)

% calculates varonthemea in vbstar

n = length(A);
E = zeros(n,length(x));
for i = 1:(n)
    E(i,:) = -nrts(i)*varontheme(A(i),B(i),C(i),D(i),x);
end
v = sum(E);
```

D.15 fplussum.m

```
function v = fplussum(sigma,mu,delta,c,alpha,beta,q,torb...
            ,lambda,lphiq,sphiq,latsphiq,nrtsh,latnrtsh,bstar,x)

% evaluates the F function plus a series

if torb == 0
    F = (1/(latsphiq*(lphiq - sphiq)))...
    - (1/(theta32p(sigma,lphiq,mu,delta,c,alpha,beta,q)-q));
elseif torb == 1
    F = (1/(latsphiq*(lphiq - sphiq)))...
    - (1/(theta52p(sigma,lphiq,mu,delta,c,alpha,beta,q)-q));
else
    F = (1/(latsphiq*(lphiq - sphiq)))...
    - (1/(beta13p(sigma,lphiq,mu,delta,c,alpha,beta,q,lambda)-q));
end

n = length(nrtsh);
A = zeros(n,length(x));
for i = 1:n
    A(i,:) = exp(-(x-bstar)*(lphiq - nrtsh(i)))/((lphiq - nrtsh(i))*latnrtsh(i));
end
v = F + sum(A);
```

D.16 doublesum2.m

```
function v = doublesum2(nrts,nrtsh,latnrts,latnrtsh,bstar,x)

% calculates the double sum in vbstar

n = length(nrts);
m = length(x);
```

```
A = zeros(n,n,m);
NRTS = zeros(n,n);
LATNRTS = zeros(n,n);
NRTSH = zeros(n,n);
LATNRTSH = zeros(n,n);
NRTS(:,1) = nrts;
LATNRTS(:,1) = latnrts;

for k = 1:n
    NRTSH(:,k) = nrtsh;
    LATNRTSH(:,k) = latnrtsh;
end

for l = 2:n
    NRTS(:,l) = circshift(nrts, [1 (l-1)]);
    LATNRTS(:,l) = circshift(latnrts, [1 (l-1)]);
end
for i = 1:m
    A(:,:,i) = (-NRTS.*(exp(x(i)*NRTS)-exp(NRTSH*x(i)...
        + bstar*(NRTS - NRTSH)))./((NRTS - NRTSH).*LATNRTSH.*LATNRTS);
end
B = sum(A,1);
C = sum(B,2);
v = squeeze(C)';
```

D.17 theta32p.m, theta52p.m, beta13p.m

This program calculates $\psi(z)$ for theta processes with $\lambda = 3/2$. Programs theta52p.m, and beta13p.m perform the same calculation for the other two types of processes. These are not shown here as the code is nearly identical.

```
function v = theta32(sigma,z,mu,delta,c,alpha,beta,q)

% Calculates the Laplace exponent of a theta process with
% lambda = 3/2. For the unperturbed process set delta = 0.
```

```
v = (1/2)*(sigma^2)*(z^2) + (mu-delta)*z - c*((alpha + z/beta)^(1/2))...
    *coth(pi*((alpha + z/beta)^(1/2))) + c*(alpha^(1/2))*coth(pi*(alpha^(1/2)));
```

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