Approximating Lévy processes with completely monotone jumps

Daniel Hackmann

Department of Mathematics and Statistics
York University
Toronto, Canada

July 7, 2014

Joint work with Alexey Kuznetsov.

Definitions and notations

A Lévy process X is specified by the triple (a, σ^2, Π), where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(\mathrm{d}x)$ satisfies $\int_{\mathbb{R}} \min(1, x^2)\Pi(\mathrm{d}x) < \infty$.

The Laplace exponent $\psi(z)$ is defined as

$$\mathbb{E}\left[e^{zX_t}\right] = e^{t\psi(z)}, \quad \mathrm{Re}(z) = 0.$$

The Lévy-Khintchine representation for $\psi(z)$ is

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + \int\limits_{\mathbb{R}} \left(e^{zx} - 1 - zx\mathbf{1}_{\{|x|<1\}}\right)\Pi(\mathrm{d}x).$$

Definitions and notations

- We define the supremum $\overline{X}_t = \sup\{X_s \ : \ 0 \le s \le t\}$ and similarly for the infimum $\underline{X}_t$;
- $\mathrm{e}(q)$ denotes an exponential random variable (with mean $1/q$), independent of X;
- The Wiener-Hopf factors are defined as $\phi_q^+(z) = \mathbb{E}[\exp(-z\overline{X}_{\mathrm{e}(q)})]$ and $\phi_q^-(z) = \mathbb{E}[\exp(-z\underline{X}_{\mathrm{e}(q)})]$;

Wiener-Hopf factorization

- $X_{e(q)} - \overline{X}_{e(q)}$ is independent of $\overline{X}_{e(q)}$ and has the same distribution as $\underline{X}_{e(q)}$.

Wiener-Hopf factorization

- $X_{e(q)} - \overline{X}_{e(q)}$ is independent of $\overline{X}_{e(q)}$ and has the same distribution as $\underline{X}_{e(q)}$.
- $$\frac{q}{q-\psi(z)} = \phi_q^+(-z)\phi_q^-(z),$$

  since

  $$\frac{q}{q-\psi(z)} = \mathbb{E}\left[e^{zX_{e(q)}}\right] = \mathbb{E}\left[e^{z(X_{e(q)}-\overline{X}_{e(q)})+z\overline{X}_{e(q)}}\right]$$

Wiener-Hopf factorization

- $X_{e(q)} - \overline{X}_{e(q)}$ is independent of $\overline{X}_{e(q)}$ and has the same distribution as $\underline{X}_{e(q)}$.
- $$\frac{q}{q-\psi(z)} = \phi_q^+(-z)\phi_q^-(z),$$
  since
  $$\frac{q}{q-\psi(z)} = \mathbb{E}\left[e^{zX_{e(q)}}\right] = \mathbb{E}\left[e^{z(X_{e(q)}-\overline{X}_{e(q)})+z\overline{X}_{e(q)}}\right]$$
- If we can factorize $q/(q-\psi(z))$ as a product of two functions $f^{\pm}(z)$, such that $f^+(z)$ {resp. $f^-(z)$} is analytic and zero-free in the half-plane $\text{Re}(z) > 0$ {resp. $\text{Re}(z) < 0$}, (plus some growth conditions) - then we can identify $\phi_q^{\pm}(z) = f^{\pm}(z)$.

Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).

Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).
- In order to price European options we need to have explicit formulas for the Laplace exponent $\psi(z)$

Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).
- In order to price European options we need to have explicit formulas for the Laplace exponent $\psi(z)$

Carr, P. and Madan, D.,
Option pricing and the fast Fourier transform.
Quantitative Finance, 2(4):61–73, 1999.

Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).
- In order to price European options we need to have explicit formulas for the Laplace exponent $\psi(z)$

Carr, P. and Madan, D.,
Option pricing and the fast Fourier transform.
Quantitative Finance, 2(4):61–73, 1999.

- The basic building blocks for pricing various exotic options (barrier/lookback/Asian/etc.) are the distributions of $\underline{X}_{e(q)}$ and $\overline{X}_{e(q)}$, so we need explicit Wiener-Hopf factors.

Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).
- In order to price European options we need to have explicit formulas for the Laplace exponent $\psi(z)$

Carr, P. and Madan, D.,
Option pricing and the fast Fourier transform.
Quantitative Finance, 2(4):61–73, 1999.

- The basic building blocks for pricing various exotic options (barrier/lookback/Asian/etc.) are the distributions of $\underline{X}_{\mathrm{e}(q)}$ and $\overline{X}_{\mathrm{e}(q)}$, so we need explicit Wiener-Hopf factors.

Jeannin, M. and Pistorius, M.,
A transform approach to calculate prices and greeks of barrier options driven by a class of Lévy processes.
Quantitative Finance, 10:629–644, 2010.

Popular processes in mathematical finance

	Variance Gamma (VG)	**Normal Inverse Gaussian (NIG)**	**Generalized Tempered Stable (CGMY or KoBol)**	**Hyper-exponential**
Activity	Infinite	Infinite	Parameter dependent	Finite
Variation	Finite	Infinite	Parameter dependent	Finite
WHF	No closed form	No closed form	No closed form	Rational function

E.g.: The Laplace exponent of the VG process:

$$\psi(z) = z\gamma + \frac{1}{k} \ln\left(1 - \frac{\sigma^2 k}{2} z^2 - \theta k z\right).$$

Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$, $n = 0, 1, 2, \dots$.

Definition

A Lévy process has completely monotone jumps, if $\Pi(\mathrm{d}x)$ is absolutely continuous with density $\pi(x)$, and $\pi(x)$ and $\pi(-x)$ are completely monotone for $x \in (0, \infty)$.

Theorem

The jump density of a process X is completely monotone if and only if $\overline{X}_{\mathrm{e}(q)}$ and $\underline{X}_{\mathrm{e}(q)}$ are mixtures of exponentials.

L.C.G. Rogers.
Wiener-Hopf factorization of diffusions and Lévy processes.
Proc. London Math. Soc., 47(3):177–191, 1983.

Hyperexponential processes

- The density of the Lévy measure is

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^{N} a_i \rho_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i e^{\hat{\rho}_i x},$$

where all the coefficients are positive.

- The Laplace exponent is a rational function

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + z \sum_{i=1}^{N} \frac{a_i}{\rho_i - z} - z \sum_{i=1}^{\hat{N}} \frac{\hat{a}_i}{\hat{\rho}_i + z}.$$

Hyperexponential processes

Assume $\sigma > 0$.

- The Wiener-Hopf factors are given by

$$\phi_q^+(z) = \frac{1}{1+\frac{z}{\zeta_1}} \prod_{i=1}^{N} \frac{1+\frac{z}{\rho_i}}{1+\frac{z}{\zeta_{i+1}}}, \quad \phi_q^-(z) = \frac{1}{1+\frac{z}{\hat{\zeta}_1}} \prod_{i=1}^{\hat{N}} \frac{1+\frac{z}{\hat{\rho}_i}}{1+\frac{z}{\hat{\zeta}_{i+1}}},$$

  where ζ_i and $\hat{\zeta}_i$ are the (real) solutions to $\psi(z) = q$.
- The distribution of $\overline{X}_{\mathrm{e}(q)}$ is a mixture of exponentials

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\overline{X}_{\mathrm{e}(q)} \le x) = \sum_{i=1}^{N+1} c_i \zeta_i e^{-\zeta_i x},$$

  where $c_i > 0$ and $\sum c_i = 1$, and similarly for $\underline{X}_{\mathrm{e}(q)}$.

Summary

- Processes with hyper-exponential jumps are great to work with, but...

Summary

- Processes with hyper-exponential jumps are great to work with, but...
- we have a problem: we can't have jumps of infinite activity/infinite variation.

Summary

- Processes with hyper-exponential jumps are great to work with, but...
- we have a problem: we can't have jumps of infinite activity/infinite variation.
- The other processes are completely montone and have infinite activity, but we do not have closed form expressions for the Wiener-Hopf factors.

Summary

- Processes with hyper-exponential jumps are great to work with, but...
- we have a problem: we can't have jumps of infinite activity/infinite variation.
- The other processes are completely montone and have infinite activity, but we do not have closed form expressions for the Wiener-Hopf factors.
- **How do we approximate a general Lévy process with completely monotone jumps by a hyperexponential process?**

Outline

Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.

Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.

Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.
- Thus we have two problems:

Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.
- Thus we have two problems:

  (1) Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,

Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.
- Thus we have two problems:

(1) Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,

(2) Show that $\tilde{\psi}(z)$ is itself a Laplace exponent of a Lévy process.

Padé approximation

Definition

Let f be a function with a power series representation $f(z) = \sum_{i=0}^{\infty} c_i z^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P) \leq m$, $\deg(Q) \leq n$, $Q_n(0) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = c_0 + c_1 z + \cdots + c_{m+n} z^{m+n} + O(z^{m+n+1}), \quad z \to 0,$$

then we say that $f^{[m/n]}(z) := P_m(z)/Q_n(z)$ is the $[m/n]$ *Padé approximant of* f.

A simple example of Padé approximations

m \ n	0	1	2	3
0	$\frac{1}{1}$	$\frac{1}{1-z}$	$\frac{1}{1-z+\frac{1}{2}z^2}$	$\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$
1	$\frac{1+z}{1}$	$\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$	$\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\frac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$
2	$\frac{1+z+\frac{1}{2}z^2}{1}$	$\frac{1+\frac{2}{3}z+\frac{1}{6}z^2}{1-\frac{1}{3}z}$	$\frac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$	$\frac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$
3	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}{1}$	$\frac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$	$\frac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$	$\frac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$
4	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4}{1}$	$\frac{1+\frac{4}{5}z+\frac{3}{10}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-\frac{1}{5}z}$	$\frac{1+\frac{2}{3}z+\frac{1}{5}z^2+\frac{1}{30}z^3+\frac{1}{360}z^4}{1-\frac{1}{3}z+\frac{1}{30}z^2}$	$\frac{1+\frac{4}{7}z+\frac{1}{7}z^2+\frac{2}{105}z^3+\frac{1}{840}z^4}{1-\frac{3}{7}z+\frac{1}{14}z^2-\frac{1}{210}z^3}$

Figure: The initial part of the Padé table for e^z

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$
- For each n we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of ν, i.e.

$$\int_{[a,b]} x^j \nu(\mathrm{d}x) = \sum_{i}^{n} x_i^j w_i, \quad , \text{for } j = 1, \dots, 2n-1.$$

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$
- For each n we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of ν, i.e.

$$\int_{[a,b]} x^j \nu(\mathrm{d}x) = \sum_i^n x_i^j w_i, \quad , \text{for } j = 1, \ldots, 2n - 1.$$

- The points x_i and w_i are the nodes and weights of the Gaussian quadrature.

Gaussian quadrature and orthogonal polynomials

- $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the measure $\nu(\mathrm{d}x)$: $\deg(p_n) = n$ and

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x)p_m(x)\nu(dx) = d_n\delta_{n,m}$$

Gaussian quadrature and orthogonal polynomials

- $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the measure $\nu(\mathrm{d}x)$: $\deg(p_n) = n$ and

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = d_n \delta_{n,m}$$

- The nodes of the Gaussian quadrature $\tilde{\nu}_n$ are the zeros of p_n and the weights may be calculated from p_{n-1}, p_n.

Gaussian quadrature and orthogonal polynomials

- $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the measure $\nu(\mathrm{d}x)$: $\deg(p_n) = n$ and

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = d_n \delta_{n,m}$$

- The nodes of the Gaussian quadrature $\tilde{\nu}_n$ are the zeros of p_n and the weights may be calculated from p_{n-1}, p_n.

G. Szegö.
Orthogonal Polynomials.
Amer. Math. Soc., Providence, RI, 4 edition, 1975.

Main theorem (two-sided case)

Assumption: The Lévy measure $\Pi(\mathrm{d}x)$ is absolutely continuous, and its density $\pi(x)$ is completely monotone and decreases exponentially fast as $x \to \pm\infty$.

Using Bernstein's theorem, we see that there exists a positive measure μ, with support in $\mathbb{R} \setminus \{0\}$, such that for all $x \in \mathbb{R}$

$$\pi(x) = \mathbb{I}_{\{x>0\}} \int_{(0,\infty)} e^{-ux} \mu(\mathrm{d}u) + \mathbb{I}_{\{x<0\}} \int_{(-\infty,0)} e^{-ux} \mu(\mathrm{d}u). \qquad (1)$$

We denote

$$\mu^*(A) = \mu(\{v \in \mathbb{R} \ : \ v^{-1} \in A\}).$$

Then $|v|^3 \mu^*(\mathrm{d}v)$ is a finite measure, with bounded support.

Main theorem (two-sided case)

Assume that $\sigma = 0$. Let $\{x_i\}_{1\le i\le n}$ and $\{w_i\}_{1\le i\le n}$ be the nodes and the weights of the Gaussian quadrature of order n with respect to the measure $|v|^3\mu^*(\mathrm{d}v)$. We define

$$\psi_n(z) := az + z^2 \sum_{i=1}^{n} \frac{w_i}{1 - zx_i}.$$

Theorem

(i) *The function $\psi_n(z)$ is the $[n+1/n]$ Padé approximant of $\psi(z)$.*

(ii) *The function $\psi_n(z)$ is the Laplace exponent of a hyperexponential process $X^{(n)}$ having the characteristic triple $(a, \sigma_n^2, \pi_n)_{h\equiv x}$, where*

Main theorem (two-sided case)

Theorem

(ii)

$$\pi_n(x) := \begin{cases} \sum_{1\le i\le n\,:\,x_i<0} w_i |x_i|^{-3} e^{-\frac{x}{x_i}}, & \text{if } x<0, \\ \sum_{1\le i\le n\,:\,x_i>0} w_i x_i^{-3} e^{-\frac{x}{x_i}}, & \text{if } x>0. \end{cases}$$

(iii) *The random variables* $X_1^{(n)}$ *and* X_1 *satisfy* $\mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j]$ *for* $1 \le j \le 2n+1$.

Convergence

Theorem

For any compact set $A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$ *there exist* $c_1 = c_1(A) > 0$ *and* $c_2 = c_2(A) > 0$ *such that for all* $z \in A$ *and all* $n \geq 1$

$$|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.$$

One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.

One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.
- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.

One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.
- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.
- There exist explicit formulas for a number of important examples: In the VG case we have $\psi^{[n/n]}(z) = P_n(z)/Q_n(z)$, where

$$P_n(z) = 2\sum_{j=0}^{n} \binom{n}{j}^2 \left[H_{n-j} - H_j\right](1-z)^j, Q_n(z) = z^n P_n\left(\tfrac{2}{z} - 1\right).$$

and $H_j := 1 + 1/2 + \cdots + 1/j$.

How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process

How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process
- The Lévy-Khintchine formula + Fubini's theorem + change of variables give us

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + z^2 \int_{[a,b]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz},$$

where $a < 0 < b$.

How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process
- The Lévy-Khintchine formula + Fubini's theorem + change of variables give us

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + z^2 \int_{[a,b]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz},$$

where $a < 0 < b$.

- $\psi(z)$ is closely related to *Stieltjes functions*:

$$f(z) := \int_{[0,R^{-1}]} \frac{\nu(\mathrm{d}u)}{1 + zu}$$

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$.

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n-1$.
- The poles of $f^{[m/n]}(z)$ are simple, real and less than $-R$, and have positive residues.

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n-1$.
- The poles of $f^{[m/n]}(z)$ are simple, real and less than $-R$, and have positive residues.
-

$$f^{[n-1/n]}(z) = \frac{(-z)^{n-1}q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1+x_i z}.$$

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n-1$.
- The poles of $f^{[m/n]}(z)$ are simple, real and less than $-R$, and have positive residues.
- $$f^{[n-1/n]}(z) = \frac{(-z)^{n-1} q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1+x_i z}.$$
- Plus convergence results

Outline

Comparing the Lévy density

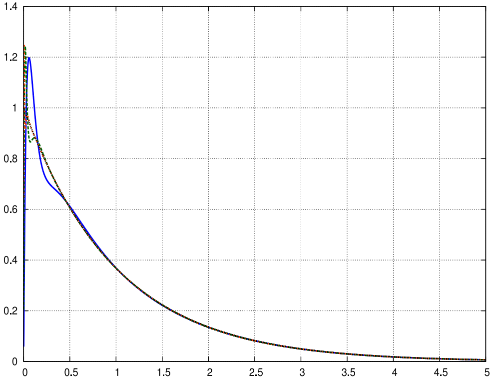

Figure: The graph of $x\pi(x)$ (black curve) and $x\pi^{[n/n]}(x)$, where $\pi(x) = \exp(-x)/x$ is the Lévy density of the Gamma subordinator, and $\pi^{[n/n]}(x)$ is the Lévy density corresponding to $\psi^{[n/n]}(z)$ Padé approximation. Blue, green and red curves correspond to $n \in \{5, 10, 20\}$.

Comparing the Lévy density

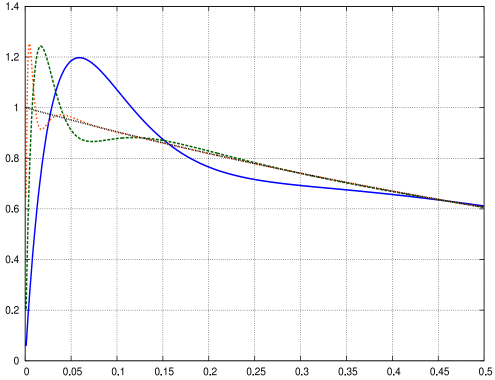

Figure: The graph of $x\pi(x)$ (black curve) and $x\pi^{[n/n]}(x)$, where $\pi(x) = \exp(-x)/x$ is the Lévy density of the Gamma subordinator, and $\pi^{[n/n]}(x)$ is the Lévy density corresponding to $\psi^{[n/n]}(z)$ Padé approximation. Blue, green and red curves correspond to $n \in \{5, 10, 20\}$.

Comparing the CDF of X_1

$\epsilon_{n,k}(1)$	$k=0$	$k=1$	$k=2$
$n=5$	$1.1e-2$	$1.1e-2$	$8.8e-3$
$n=10$	$2.8e-3$	$3.4e-3$	$2.8e-3$
$n=15$	$1.3e-3$	$1.6e-3$	$1.4e-3$
$n=20$	$7.5e-4$	$9.3e-4$	$8.1e-4$

Table: The values of $\epsilon_{n,k}(t) := \max_{x\geq 0} |\mathbb{P}(X_t \leq x) - \mathbb{P}(X_t^{(n,k)} \leq x)|$, where X is the Gamma process with $\psi(z) = -\ln(1-z)$ and the process $X^{(n,k)}$ has Laplace exponent $\psi^{[n+k/n]}$.

Comparing the CDF of X_2

$\epsilon_{n,k}(2)$	$k=0$	$k=1$	$k=2$
$n=5$	$3.3e-4$	$3.2e-4$	$5.4e-4$
$n=10$	$2.6e-5$	$2.8e-5$	$5.6e-5$
$n=15$	$5.4e-6$	$6.4e-6$	$1.3e-5$
$n=20$	$1.8e-6$	$2.1e-6$	$4.6e-6$

Table: The values of $\epsilon_{n,k}(t) := \max_{x\geq 0} |\mathbb{P}(X_t \leq x) - \mathbb{P}(X_t^{(n,k)} \leq x)|$, where X is the Gamma process with $\psi(z) = -\ln(1-z)$ and the process $X^{(n,k)}$ has Laplace exponent $\psi^{[n+k/n]}$.

Math Finance applications

We will work with the following two processes: the VG process V defined by the Laplace exponent

$$\psi(z) = \mu z - \frac{1}{\nu}\log\left(1 - \frac{z}{a}\right) - \frac{1}{\nu}\log\left(1 + \frac{z}{\hat{a}}\right),$$

and parameters

$$(a, \hat{a}, \nu) = (21.8735,\ 56.4414,\ 0.20),$$

and the CGMY process Z defined by the Laplace exponent

$$\psi(z) = \mu z + C\Gamma(-Y)\left[(M - z)^Y - M^Y + (G + z)^Y - G^Y\right],$$

and parameters

$$(C, G, M, Y) = (1,\ 8.8,\ 14.5,\ 1.2).$$

European call option

	two-sided $[2N+1/2N]$	one-sided $[N/N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-1.58e-2	9.12e-2	7.02e-3	-3.02e-2
$N=2$	1.66e-3	-6.16e-3	4.80e-3	-7.82e-4
$N=3$	6.20e-4	-1.28e-3	-4.32e-5	6.78e-4
$N=4$	1.25e-4	1.88e-4	-1.98e-4	9.81e-5
$N=5$	-7.19e-5	8.82e-5	-2.62e-5	-2.40e-5
$N=7$	4.34e-6	-8.48e-6	5.82e-6	-1.71e-6
$N=9$	-7.72e-8	3.31e-7	-6.99e-7	7.35e-7
$N=12$	4.85e-7	-1.81e-8	4.97e-8	-6.10e-8
$N=15$	-8.56e-8	-1.37e-9	-3.31e-9	6.06e-9

Table: The error in computing the price of the European call option for the VG V-model. Initial stock price is $S_0 = 100$, strike price $K = 100$, maturity $T = 0.25$ and interest rate $r = 0.04$. The benchmark price is 2.5002779303.

European call option

	two-sided $[2N+1/2N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-2.75e-2	1.93e-2	-3.72e-3
$N=2$	-4.86e-6	-4.19e-6	9.5e-5
$N=3$	4.80e-7	-1.48e-5	-2.54e-7
$N=4$	2.9e-8	6.41e-7	-1.55e-7
$N=5$	1.14e-9	5.58e-9	6.95e-9

Table: The error in computing the price of the European call option for the CGMY Z-model. The benchmark price is 11.9207826467.

Asian option

Asian call option

$$C(S_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\int_0^T S_u \mathrm{d}u - K\right)^+\right].$$

We set the parameters $S_0 = 100$, $r = 0.03$, $T = 1$, $K = 90$ for the VG process and $K = 110$ for the CGMY process.

	two-sided $[2N+1/2N]$	one-sided $[N/N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N = 1$	-1.87e-3	1.01e-3	-1.82e-3	9.88e-4
$N = 2$	9.49e-5	2.89e-4	-6.33e-5	3.27e-5
$N = 3$	1.30e-6	8.85e-6	-4.24e-6	3.99e-6
$N = 4$	-2.83e-6	1.07e-6	-1.36e-6	3.16e-7
$N = 5$	-1.11e-7	-2.48e-8	-5.91e-7	-3.81e-7

Table: The error in computing the price of the Asian option for the VG V-model. The benchmark price is 11.188589 (calculated using the [91/90] two-sided approximation).

Asian option

	two-sided $[2N+1/2N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	1.88e-4	7.42e-4	-1.19e-3
$N=2$	4.03e-6	9.05e-5	5.39e-6
$N=3$	-3.58e-7	-2.64e-6	7.93e-8
$N=4$	-3.88e-7	-1.01e-7	-1.21e-7
$N=5$	-5.26e-7	-2.47e-7	-2.49e-7

Table: The error in computing the price of the Asian option for the CGMY Z-model. The benchmark price is 9.959300 (calculated using the [91/90] two-sided approximation).

Barrier option

Down-and-out barrier put option:

$$D(S_0, K, B, T) := e^{-rT}\mathbb{E}\left[(K - S_T)^+ \mathbf{1}_{\{S_t > B \text{ for } 0 \le t \le T\}}\right],$$

We calculate barrier option prices for the process V, for four values $S_0 \in \{81, 91, 101, 111\}$ and with other parameters given by $K = 100$, $B = 80$, $r = 0.04879$ and $T = 0.5$

	$S_0 = 81$	$S_0 = 91$	$S_0 = 101$	$S_0 = 111$
Benchmark	3.39880	7.38668	1.40351	0.04280
$N = 2$	3.44551	7.39225	1.40527	0.04233
$N = 4$	3.40209	7.38957	1.40329	0.04258
$N = 6$	3.39910	7.38939	1.40332	0.04258
$N = 8$	3.39856	7.38936	1.40332	0.04258
$N = 10$	3.39853	7.38936	1.40332	0.04258

Table: Barrier option prices calculated for the VG process V-model. Benchmark prices obtained from "Fast and accurate pricing of barrier options under Lévy processes" by Kudryavtsev and Levendorskiĭ

References:

- D. Hackmann and A. Kuznetsov (2014)
  "Approximating Lévy processes with completely monotone jumps", arXiv:1404.0597.
- G. D. Allen, C. K. Chui, W. R. Madych, F. J. Narcowich, and P. W. Smith.
  Padé approximation of Stieltjes series.
  Journal of approximation theory, 14:302–316, 1975.
- S. G. Baker and P. Graves-Morris.
  Padé Approximants, volume 1.
  Cambridge University Press, Cambridge–New York, 2 edition, 1996.

www.danhackmann.com