Analytical methods for Lévy processes with applications to finance, Part I

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1 Introduction

2 Lévy processes: A quick introduction

3 The Wiener-Hopf factorization

4 The exponential functional

5 Applications of the Wiener-Hopf factors, and the exponential functional

6 Some examples of Lévy processes

Overview

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- Introduce Lévy processes.
- Introduce two important theoretical objects: the Wiener-Hopf factors and the exponential functional.
- Discuss two analytical techniques to determine these objects, and give two key applications in finance: pricing of Asian options and barrier options.
- Give some examples of popular processes, and consider their utility if we intend to work with the Wiener-Hopf factorization and the exponential functional.
- Give some examples for which it is relatively easy to work with the Wiener-Hopf factors and the exponential functional (analytically tractable families).

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Definition

A Lévy process is an $\mathbb{R}$-valued stochastic process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that possesses the following properties:

(i) The paths of X are right continuous with left limits $\mathbb{P}$-a.s.

(ii) $X_0 = 0$ $\mathbb{P}$-a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

Applications

Lévy processes are popular because they are general enough to represent real world phenomena, but tractable enough so that we may obtain meaningful results.

We find applications of Lévy processes in many fields, the natural sciences, operations management, actuarial science, and, of course, mathematical finance.

Basic examples

Let X be a scaled Brownian motion with drift

$$X_t := at + \sigma B_t$$

where $a \in \mathbb{R}$ and $\sigma > 0$ and B_t is a standard Brownian motion. Computing the moment generating function we get

$$\mathbb{E}[e^{zX_t}] = e^{t(az + \frac{\sigma^2}{2} z^2)}.$$

Now we define

$$\psi_X(z) := \frac{1}{t} \log \mathbb{E}[e^{zX_t}] = az + \frac{\sigma^2}{2} z^2,$$

which we will call the Laplace exponent of X.

Basic examples

Let N be a Poisson process, i.e. a Lévy process which satisfies

$$\mathbb{P}(N_t = x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}, \quad x \in \mathbb{N}.$$

Now let $\{\xi_n\}_{n\geq 1}$ be a collection of i.i.d. random variables, also independent of N, with distribution $F(\mathrm{d}x)$. Then the process

$$Y_t := \sum_{n=1}^{N_t} \xi_n$$

is a Lévy process which is called a compound Poisson process.

Basic examples

By conditioning on N we can derive the moment generating function of Y which has the form

$$\mathbb{E}[e^{zY_t}] = e^{t\lambda \int_{\mathbb{R}} (e^{zx}-1)F(\mathrm{d}x)}, \quad z \in i\mathbb{R}.$$

Again, we can calculate the Laplace exponent of Y:

$$\psi_Y(z) := \frac{1}{t} \log \mathbb{E}[e^{zY_t}] = \lambda \int_{\mathbb{R}} (e^{zx} - 1)F(\mathrm{d}x).$$

Basic examples

Now if X and Y are independent, then $Z = X + Y$ is again a Lévy process. It is easy to see that Z will have Laplace exponent

$$\psi_Z(z) = az + \frac{\sigma^2}{2} z^2 + \lambda \int_{\mathbb{R}} (e^{zx} - 1) F(\mathrm{d}x), \quad z \in i\mathbb{R}.$$

Observations:

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- The rate parameter $\mathbb{E}[N_1] = \lambda$ controls the frequency of the jumps. Since λ always remains finite, so do the expected number of jumps (finite activity).
- Let $\Pi(\mathrm{d}x) = \lambda F(\mathrm{d}x)$. Does replacing $\Pi(\mathrm{d}x)$ by an infinite measure make sense (infinite activity)?

Lévy-Khintchine formula

The short answer is "yes". Specifically, for any triple (a, σ^2, Π) where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and $\Pi(\mathrm{d}x)$ is a measure on $\mathbb{R}\backslash\{0\}$ which satisfies

$$\int_{\mathbb{R}\backslash\{0\}} \min(1, x^2)\Pi(\mathrm{d}x) < \infty,$$

the function

$$\psi(z) = az + \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}\backslash\{0\}} (e^{zx} - 1 - zx\mathbb{I}(|x| < 1))\Pi(\mathrm{d}x), \quad z \in i\mathbb{R}$$

is the Laplace exponent of a Lévy process. Conversely, the Laplace exponent of any Lévy process has the form $\psi(z)$ with (a, σ^2, Π) satisfying the above conditions.

Interpretation

Suppose $\psi(z)$ is the Laplace exponent of a Lévy process and $S := \mathbb{R}\backslash(-1,1)$. Due to the integrability condition on $\Pi(\mathrm{d}x)$ we must have $0 < \lambda := \Pi(S) < \infty$ so that

$$F(\mathrm{d}x) := \frac{\Pi(\mathrm{d}x)}{\lambda}$$

is a probability distribution on S. Accordingly, we can write

$$\psi(z) = \overbrace{az + \frac{\sigma^2}{2}z^2}^{\psi_X(z)} + \overbrace{\lambda \int_S (e^{zx} - 1)F(\mathrm{d}x)}^{\psi_Y(z)} \\ + \underbrace{\int_{|x|<1} (e^{zx} - 1 - zx)\Pi(\mathrm{d}x)}_{?}.$$

Interpretation

We can repeat this. Let $S_n := 2^{-(n+1)} \leq |x| < 2^{-n}$ for $n \in \{0\} \cup \mathbb{N}$. Then $0 < \lambda_n := \Pi(S_n) < \infty$ so that

$$F_n(\mathrm{d}x) := \frac{\Pi(\mathrm{d}x)}{\lambda_n}$$

is a probability distribution on S_n. Therefore,

$$\int_{|x|<1} (e^{zx} - 1 - zx)\Pi(\mathrm{d}x) = \sum_{n \geq 0} \left[\lambda_n \int_{S_n} (e^{zx} - 1) F_n(\mathrm{d}x) - z\lambda_n \int_{S_n} x F_n(\mathrm{d}x) \right]$$

Interpretation

The last expression is the Laplace exponent of an infinite sum of independent Poisson processes compensated by a linear drift. One can show carefully that this is, in fact, the Laplace exponent of a Lévy process. Details in:

A.E. Kyprianou.
Introductory Lectures on Fluctuations of Lévy Processes with Applications.
Springer–Verlag, Berlin–Heidelberg–New York, 1 edition, 2006.

J. Bertoin.
Lévy processes.
Cambridge University Press, Cambridge–Cape Town–Madrid–Port Melbourne–New York, 1996.

K. Sato.
Lévy processes and infinitely divisible distributions.
Cambridge University Press, Cambridge–Cape Town–Madrid–Port Melbourne–New York, 1999.

Identification of sample paths

The Laplace exponent uniquely determines the Lévy process (up to identity in law) and is very important both for theory and applications. One direct application: we can determine the behaviour of the sample paths from the triple (a, σ^2, Π):

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- If $\sigma = 0$, $\Pi(\mathbb{R}\backslash\{0\}) = \infty$, and $\int_{\mathbb{R}\backslash\{0\}} \min(1, |x|)\Pi(\mathrm{d}x) < \infty$ then we have infinite jump activity but the sample paths have finite total variation.

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- If $\sigma = 0$, $\Pi(\mathbb{R}\backslash\{0\}) = \infty$, and $\int_{\mathbb{R}\backslash\{0\}} \min(1, |x|)\Pi(\mathrm{d}x) = \infty$ the sample paths have infinite total variation.

One-sided processes

Processes which are almost surely increasing are called *subordinators*. Processes which are not subordinators but have no negative jumps are called *spectrally positive*.

Applications: random time (subordination), insurance claims, workload models

Infinite divisibility

From the Lévy-Khintchine formula (or from the definition of a Lévy process) have

$$\mathbb{E}[e^{zX_1}] = \left(e^{\frac{1}{n}\psi(z)}\right)^n = \left(\mathbb{E}[e^{zX_{1/n}}]\right)^n, \quad z \in i\mathbb{R},\ n \in \mathbb{N}$$

equivalently that

$$X_1 \overset{d}{=} X_{1/n,1} + \ldots + X_{1/n,n},$$

where the $\{X_{1/n,i}\}_{1 \le i \le n}$ are independent and distributed like $X_{1/n}$. That is, X_1 is *infinitely divisible*. The converse is also true: for each infinitely divisible random variable ξ we can construct a Lévy process such that $X_1 \overset{d}{=} \xi$ (Lévy-Khintchine formula for inf. div. r.v.) .

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Some notation

Define

$$S_t := \sup_{0 \le s \le t} X_t, \quad \text{and} \quad I_t := \inf_{0 \le s \le t} X_t,$$

and let $\mathbf{e}(q)$ denote an exponential random variable, independent of X, which has mean q^{-1}.

Further denote $S_q := S_{\mathrm{e}(q)}$ and $I_q := I_{\mathrm{e}(q)}$.

Finally, let

$$\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}, \quad \text{and} \quad \bar{\mathbb{C}}^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\},$$

with $\mathbb{C}^-$ and $\bar{\mathbb{C}}^-$ defined analogously.

The Wiener-Hopf factorization

- The *Wiener-Hopf factors* are defined as $\phi_q^+(z) := \mathbb{E}[\exp(zS_q)]$ and $\phi_q^-(z) := \mathbb{E}[\exp(zI_q)]$.

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- $X_{\mathrm{e}(q)} - S_q$ is independent of S_q and has the same distribution as I_q.
- Therefore, we have the important identity

$$\frac{q}{q-\psi(z)} = \phi_q^+(z)\phi_q^-(z), \quad z \in i\mathbb{R},$$

since

$$\frac{q}{q-\psi(z)} = \mathbb{E}\left[e^{zX_{\mathrm{e}(q)}}\right] = \mathbb{E}\left[e^{z(X_{\mathrm{e}(q)}-S_q)+zS_q}\right].$$

Explicit examples?

There are few examples of processes for which we know the Wiener-Hopf factorization explicitly, although the number of cases has grown in the last 10 years – some examples to follow. One way we can derive the WHF for a process:

Theorem (Kuznetsov, 2011)

Assume there exist two functions $f^+(z)$ and $f^-(z)$ such that $f^\pm(0) = 1$, $f^\pm(z)$ is analytic in $\mathbb{C}^\mp$, $f^\pm(z)$ is continuous without roots in $\bar{\mathbb{C}}^\mp$, and $z^{-1}\log(f^\pm(z)) \to 0$ as $z \to \infty$, $z \in \bar{\mathbb{C}}^\mp$. If

$$\frac{q}{q - \psi(z)} = f^+(z) f^-(z), \quad z \in i\mathbb{R}$$

then $f^\pm(z) = \phi_q^\pm(z)$ for all $z \in \bar{\mathbb{C}}^\mp$.

A. Kuznetsov.
Analytic proof of Pecherskii–Rogozin identity and Wiener-Hopf factorization.
Theory Probab. Appl., 55(3):432–443, 2011.

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Definition

Let X be a Lévy process, and $\mathbf{e}(q)$ be defined as before. The random variable

$$I_{\mathbf{e}(q)} := \int_0^{\mathbf{e}(q)} e^{X_t} \mathrm{d}t$$

is known as the *exponential functional* of X. If we agree that $\mathbf{e}(0) = \infty$ then the definition extends to the $q = 0$ case provided $\lim_{t\to\infty} X_t = -\infty$.

The distribution of $I_{\mathbf{e}(q)}$

For general, i.e. two sided processes, there are very few instances where we know the distribution of $I_{\mathbf{e}(q)}$ explicitly. One possibility is to approach the problem via the Mellin transform $\mathcal{M}(I_{\mathbf{e}(q)}, z) := \mathbb{E}[I_{\mathbf{e}(q)}^{z-1}]$. The following lemma and theorem are the key to this approach:

Lemma (Maulik & Zwart, 2006 & Carmona et. al., 1997)

Let $q \geq 0$ and X be a Lévy process with Laplace exponent $\psi(z)$. If $z > 0$ and $q - \psi(z) > 0$, we have

$$\mathcal{M}(I_{e(q)}, z+1) = \frac{z}{q - \psi(z)} \mathcal{M}(I_{e(q)}, z), \tag{1}$$

where the equality is interpreted to mean that both sides can be infinite.

The distribution of $I_{\mathbf{e}(q)}$

Theorem (Kuznetsov and Pardo, 2013)

Assume that Cramér's condition is satisfied: there exists $z_0 > 0$ such that the Laplace exponent $\psi(z)$ is finite for all $z \in (0, z_0)$ and $\psi(\theta) = q$ for some $\theta \in (0, z_0)$. If a function $f(z)$ satisfies the following three properties:

(i) *$f(z)$ is analytic and zero-free in the strip $\mathrm{Re}(z) \in (0, 1+\theta)$;*

(ii) *$f(1) = 1$ and $f(z+1) = zf(z)/(q - \psi(z))$ for all $z \in (0, \theta)$; and*

(iii) *$|f(z)|^{-1} = o(\exp(2\pi|\mathrm{Im}(z)|))$ as $\mathrm{Im}(z) \to \infty$, uniformly in $\mathrm{Re}(z) \in (0, 1+\theta)$,*

then $\mathcal{M}(I_{\mathbf{e}(q)}, z) \equiv f(z)$ for $\mathrm{Re}(z) \in (0, 1+\theta)$.

Some references

P. Carmona, F. Petit, and M. Yor.
On the distribution and asymptotic results for exponential functionals of Lévy processes, pages 73–130.
Bibl. Rev. Mat. Iberoamericana, Madrid, 1997.

A. Kuznetsov and J.C Pardo.
Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes.
Acta Applicandae Mathematicae, 123(1):113 – 139, 2013.

K. Maulik and B. Zwart.
Tail asymptotics for exponential functionals of Lévy processes.
Stochastic processes and their applications, 116:156–177, 2006.

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In general

Wiener-Hopf factors: Exit problems of all kinds can often be reformulated in terms of the processes S and I, and therefore also in terms of S_q and I_q and $\phi_q^+(z)$ and $\phi_q^-(z)$.

The exponential functional: Various connections to other areas of probability. For example: Self-similar Markov processes, stationary measures of generalized Ornstein-Uhlenbeck processes, and fragmentation and branching processes. Also, a connection to pricing Asian options.

Option pricing

Let X be a Lévy process and let A denote the stock price which is given by

$$A_t := A_0 e^{X_t},$$

for $A_0 > 0$. We always use $r > 0$ to denote the risk-free rate of return and we assume that we are working with a risk neutral measure, equivalently $\psi(1) = r$. All options will have fixed expiries T and fixed strikes K.

Barrier Options: Down-and-out Put

Compute

$$D(A_0, K, B, T) := e^{-rT}\mathbb{E}\left[(K - A_T)^+ \mathbb{I}\left(\inf_{0\le t\le T} A_t > B\right)\right],$$

equivalently, compute

$$f(t) := \mathbb{E}[(k - e^{X_t})^+ \mathbb{I}(I_t > b)],$$

where $0 < B < A_0$ is the barrier, $k := K/A_0$, and $b := \log(B/A_0)$.

Barrier Options: Down-and-out Put

Consider the Laplace transform of $f(t)$

$$F(q) := q \int_{\mathbb{R}+} e^{-qt} f(t) \mathrm{d}t = \mathbb{E}[(k - e^{X_{\mathbf{e}(q)}})^+ \mathbb{I}(I_q > b)].$$

Why is this useful?

M. Jeannin and M. Pistorius.
A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes.
Quantitative Finance, 10:629–644, 2010.

Barrier Options: Down-and-out Put

Using the Wiener-Hopf factorization we may rewrite $F(q)$ as

$$\begin{aligned} F(q) &= \mathbb{E}[(k - e^{X_{\mathbf{e}(q)}})^+ \mathbb{I}(I_q > b)] \\ &= \mathbb{E}[(k - e^{S_q + I_q})^+ \mathbb{I}(I_q > b)]. \end{aligned}$$

Now, if we know the Wiener-Hopf factors explicitly, or even better, if we know the distributions of I_q and S_q we can work out a semi-explicit, or even explicit expression for $F(q)$. Lookback options can be treated similarly.

Asian call

Compute

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right],$$

equivalently compute

$$f(k,t) := \mathbb{E}\left[\left(\int_0^t e^{X_u}\mathrm{d}u - k\right)^+\right].$$

Asian call

Taking the Laplace transform we get

$$h(k,q) := q\int_{\mathbb{R}+} e^{-qt} f(k,t)\mathrm{d}t = \mathbb{E}\left[\left(\int_0^{\mathbf{e}(q)} e^{X_t}\mathrm{d}t - k\right)^+\right].$$

Why is this useful?

If the distribution of $I_{\mathbf{e}(q)}$ is known, and tractable enough, then we can determine $h(k,q)$ explicitly.

M. Yor and H. Geman.
Bessel processes, Asian options, and perpetuities.
Mathematical Finance, 3(4):349–375, 1993.

Asian call

If not, we can transform $h(k,q)$ again

$$\Phi(z,q) := \int_{\mathbb{R}^+} h(k,q) k^{z-1} \mathrm{d}k = \mathbb{E}\left[\int_{\mathbb{R}^+} \left(I_{\mathbf{e}(q)} - k\right)^+ k^{z-1} \mathrm{d}k\right]$$

$$= \mathbb{E}\left[\int_0^{I_{\mathbf{e}(q)}} \left(I_{\mathbf{e}(q)} - k\right) k^{z-1} \mathrm{d}k\right] = \frac{\mathbb{E}\left[I_{\mathbf{e}(q)}^{z+1}\right]}{z(z+1)} = \frac{\mathcal{M}(I_{\mathbf{e}(q)}, z+2)}{z(z+1)}.$$

If we can find an explicit expression for $\mathcal{M}(I_{\mathbf{e}(q)}, z)$, then we have an explicit expression for $\Phi(z,q)$.

N. Cai and S.G. Kou.
Pricing Asian options under a hyper-exponential jump diffusion model.
Operations Research, 60(1):64–77, 2012.

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Examples: VG process

- The density of the Lévy measure is

$$\pi(x) = \mathbb{I}(x < 0)\frac{ce^{\hat{\rho}x}}{|x|} + \mathbb{I}(x > 0)\frac{ce^{-\rho x}}{x},$$

  where c, $\hat{\rho}$, $\rho > 0$.
- The process has infinite activity jumps and finite variation.
- The Laplace exponent is

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z - c\log\left(1 + \frac{z}{\hat{\rho}}\right) - c\log\left(1 - \frac{z}{\rho}\right).$$

- The Wiener-Hopf factors nor the distribution of the exponential functional are known explicitly.

Examples: generalized tempered stable process (KoBoL, CGMY)

- The density of the Lévy measure is

$$\pi(x) = \mathbb{I}(x<0)\frac{\hat{c}}{|x|^{1+\hat{\alpha}}}e^{\hat{\rho}x} + \mathbb{I}(x>0)\frac{c}{x^{1+\alpha}}e^{-\rho x},$$

  where $\hat{c}$, $\hat{\rho}$, c, $\rho > 0$ and $\hat{\alpha}, \alpha \in (-1,2)\backslash\{0,1\}$.
- The Laplace exponent is

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \Gamma(-\hat{\alpha})\hat{c}\big((\hat{\rho}+z)^{\hat{\alpha}} - \hat{\rho}^{\hat{\alpha}}\big) + \Gamma(-\alpha)c\big((\rho - z)^{\alpha} - \rho^{\alpha}\big).$$

- The Wiener-Hopf factors nor the distribution of the exponential functional are known explicitly.

Examples: Other popular stock price models

Other popular models include Generalized Hyperbolic (GH) processes and the special case, Normal Inverse Gaussian (NIG) processes. These, and the previous models have been successfully fit to financial time series and are attractive in the sense that most have explicit transition densities and are relatively analytically tractable. Still, for none do we know the Wiener-Hopf factors nor the distribution of the exponential functional explicitly.

What are some families where we do have this information?

Analytically tractable families

- Processes with jumps of rational transform
  - Processes with phase-type jumps
  - Hyper-exponential processes
- Meromorphic processes

Example: Hyper-exponential process

The density of the Lévy measure is

fwd1 fwd2

$$\pi(x) = \mathbb{I}(x > 0) \sum_{n=1}^{N} a_n \rho_n e^{-\rho_n x} + \mathbb{I}(x < 0) \sum_{i=1}^{\hat{N}} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x},$$

where all the coefficients are positive.

Example: Hyper-exponential process

The Laplace exponent is a rational function

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + z^2 \sum_{n=1}^{\hat{N}} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n - z)},$$

and the (real) solutions ζ_n and $-\hat{\zeta}_n$ of $\psi(z) = q$ and the poles of $\psi(z)$ satisfy the important *interlacing property*

$$0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 \ldots$$
$$0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 \ldots .$$

Example: Hyper-exponential process

Assume $\sigma > 0$

- The Wiener-Hopf factors are given by

$$\phi_q^+(z) = \frac{1}{1+\frac{z}{\zeta_1}} \prod_{n=1}^{N} \frac{1-\frac{z}{\rho_n}}{1-\frac{z}{\zeta_{n+1}}}, \quad \phi_q^-(z) = \frac{1}{1+\frac{z}{\hat{\zeta}_1}} \prod_{n=1}^{\hat{N}} \frac{1+\frac{z}{\hat{\rho}_n}}{1+\frac{z}{\hat{\zeta}_{n+1}}},$$

- The distribution of S_q is a mixture of exponentials

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(S_q \le x) = \sum_{n=1}^{N+1} c_n \zeta_n e^{-\zeta_n x},$$

where $c_n > 0$ and $\sum c_n = 1$, and similarly for I_q.

Example: Hyper-exponential process

Define

$$\mathcal{G}(z) := \frac{\prod_{n=1}^{N+1} \Gamma(\zeta_k - z + 1)}{\prod_{n=1}^{N} \Gamma(\rho_n - z + 1)} \times \frac{\prod_{n=1}^{\hat{N}} \Gamma(\hat{\rho}_n + z)}{\prod_{k=1}^{\hat{N}+1} \Gamma(\hat{\zeta}_k + z)},$$

then

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = \left(\frac{\sigma^2}{2}\right)^{1-z} \times \Gamma(z) \times \frac{\mathcal{G}(z)}{\mathcal{G}(1)},$$

and

$$I_{\mathbf{e}(q)} \overset{d}{=} \frac{2}{\sigma^2} \frac{B_{(1,\hat{\zeta}_1)}}{G_{(\zeta_{N+1},1)}} \frac{\prod_{n=1}^{\hat{N}} B_{(\hat{\rho}_n+1,\hat{\zeta}_{n+1}-\hat{\rho}_n)}}{\prod_{n=1}^{N} B_{(\zeta_n,\rho_n-\zeta_n)}}.$$

Example: Hyper-exponential process

It seems like Hyper-exponential processes have everything we want, however, they have one serious disadvantage: hyper-exponential processes are necessarily finite activity processes, and for applications in finance we often want infinite activity processes, sometimes maybe even infinite variation.

Meromorphic processes

Meromorphic processes are the generalization of hyper-exponential processes. Essentially, a meromorphic process results from replacing the finite sum in the Lévy density of a hyper-exponential process by an infinite series.

recall

Everything has an "infinite" analogue which has precisely the expected form.

	Jump activity	Laplace exponent	Interlacing property	Wiener-Hopf factors	Exp. functional
Hyper-exp.	Finite	Rational	Yes	Finite product	Finite product of r.v.'s
Mero-morphic	Any	Mero-morphic	Yes	Infinite product	Infinite product of r.v.'s *

Beta family

A Beta process is a Lévy process with Lévy density

$$\pi(x) = \mathbb{I}(x < 0) c_1 \frac{e^{\alpha_1 \beta_1 x}}{(1 - e^{\beta_1 x})^{\lambda_1}} + \mathbb{I}(x > 0) c_2 \frac{e^{-\alpha_2 \beta_2 x}}{(1 - e^{-\beta_2 x})^{\lambda_2}},$$

where c_i, α_i, $\beta_i > 0$ and $\lambda_i \in (0,3)/\{1,2\}$. Choosing $\lambda_i > 1$ yields an infinite activity processes, and if $\lambda_i > 2$ then the process will also have infinite variation. The origin of its name is its Laplace exponent, which has the following closed-form expression,

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + \frac{c_1}{\beta_1} \left(\mathrm{B}\left(\alpha_1 + \frac{z}{\beta_1}, 1 - \lambda_1 \right) - \mathrm{B}(\alpha_1, 1 - \lambda_1) \right)$$
$$+ \frac{c_2}{\beta_2} \left(\mathrm{B}\left(\alpha_2 - \frac{z}{\beta_2}, 1 - \lambda_2 \right) - \mathrm{B}(\alpha_2, 1 - \lambda_2) \right).$$

Beta family (cont.)

With the help of the binomial series, one may verify that $\pi(x)$ has the form we expect, specifically, that

$$\hat{\rho}_n = \beta_1(\alpha_1 + n - 1), \quad \rho_n = \beta_2(\alpha_2 + n - 1),$$

$$\hat{a}_n = \hat{\rho}_n^{-1} c_1 \binom{n + \lambda_1 - 2}{n - 1}, \quad \text{and} \quad a_n = \rho_n^{-1} c_2 \binom{n + \lambda_2 - 2}{n - 1}.$$

Compare

Achieving a realistic *and* tractable model

Suppose we want to price barrier or Asian options using our algorithms, and we want to ensure that our model has infinite jump activity. What can we do?

1. Use meromorphic processes.
2. Use one of the popular VG, CGMY, GH, NIG models and approximate by a meromorphic process.
3. Use one of the popular VG, CGMY, GH, NIG models and approximate by a hyper-exponential process. **

Example: approximating by a Beta process

Compare the CGMY Lévy density

$$\pi(x) = \mathbb{I}(x<0) C \frac{e^{Gx}}{|x|^{1+Y}} + \mathbb{I}(x>0) C \frac{e^{-Mx}}{x^{1+Y}},$$

with the Beta family Lévy density

$$\tilde{\pi}(x) = \mathbb{I}(x<0) c_1 \frac{e^{\alpha_1 \beta_1 x}}{(1-e^{\beta_1 x})^{\lambda_1}} + \mathbb{I}(x>0) c_2 \frac{e^{-\alpha_2 \beta_2 x}}{(1-e^{-\beta_2 x})^{\lambda_2}}.$$

Set $c_1 = c_2 = C\beta^{1+Y}$, $\alpha_1 = G\beta^{-1}$, $\alpha_2 = M\beta^{-1}$, $\lambda_1 = \lambda_2 = 1+Y$, $\beta_1 = \beta_2 = \beta$.

We see as $\beta \to 0^+$ we have $\tilde{\pi}(x) \to \pi(x)$.